Finite fibers of multi-graded dense rational maps on $\mathbb{P}^3$

Nicolas Botbol Laurent Busé Marc Chardin
Fatmanur Yıldırım

Finite fibers of rational maps Ψ from $\mathbb{P}^2$ to $\mathbb{P}^3$ and also $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ have been studied in [2] by Botbol, Busé and Chardin. For both cases, they have assumed that the base locus $\mathcal{B}$ of Ψ is finite and locally complete intersection. Also for the both cases, $\mathcal{B}$ contains only points and multiple of points. Under these assumptions, they write approximation complexes of cycles $\mathcal{Z}_\bullet$ of multi-degree ν computed according to [1]. They show that for these ν values, the Hilbert function of the fiber at a point $p \in \mathbb{P}^3$, becomes Hilbert polynomial of the fiber at p. From the first arrow of $\mathcal{Z}_\bullet$ giving a family of matrices they chose one and denote it by $\mathcal{M}(\Psi)_\nu$. They study the fibers of dimension 0 and 1. They state a relation between the corank of $\mathcal{M}(\Psi)(p)_\nu$ and degree of the fiber at point p.

Our motivation is to compute the distance from a point $p \in \mathbb{R}^3$ to an algebraic rational surface $\mathcal{S} \in \mathbb{R}^3$. For that purpose, firstly, from a parametrization of $\mathcal{S}$, we construct a parametrization of normal lines to $\mathcal{S}$, namely ψ from $\mathbb{R}^3$ into $\mathbb{R}^3$. Secondly, we homogenize ψ in $\mathbb{P}^2 \times \mathbb{P}^1$ or possibly in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and we obtain a multi-graded multi homogeneous rational map Ψ into $\mathbb{P}^3$.

Our work is an extension of [2]. We present a new method to study the finite fibers of dense multi-graded rational maps Ψ from $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$. Different from the previous work [2], in this case the base locus $\mathcal{B}$ of Ψ has degree one components, i.e it contains curves. This was the main difficulty to handle and we needed additional hypothesis to control the fiber over a point $p \in \mathbb{P}^3$. For the multi-degree ν computed again according to [1], we show that Hilbert function of the fiber at point $p \in \mathbb{P}^3$ becomes Hilbert polynomial at p which is equal to the corank of $\mathcal{M}(\Psi)(p)_\nu$. Moreover, at these multi-degree ν, the corank of $\mathcal{M}(\Psi)(p)_\nu$ is equal to degree of the fiber at p and it gives us the number of the orthogonal projections of p onto the surface $\mathcal{S}$ under our construction.

References

[1] Nicolás Botbol. The implicit equation of a multigraded hypersurface. *Journal of Algebra*, 348(1):381 – 401, 2011.

[2] Nicolàs Botbol, Laurent Busé, and Marc Chardin. Fitting ideals and multiple points of surface parameterizations. *Journal of Algebra*, 420(Supplement C):486 – 508, 2014.