Fast and rigorous arbitrary-precision evaluation of Legendre polynomials and Gauss-Legendre quadrature nodes

Fredrik Johansson* Marc Mezzarobba†

Gauss-Legendre quadrature requires a nearly minimal number of evaluation points to achieve a given accuracy for numerical integration of functions that are well-approximated by polynomials. However, other quadrature rules (such as the Clenshaw-Curtis and double exponential formulas) have often been favored in high-precision computations due to the cost of generating the quadrature nodes, even in cases where those rules require more evaluation points.

We describe an efficient strategy for rigorous arbitrary-precision evaluation of Legendre polynomials on the unit interval and its application in the generation of Gauss-Legendre quadrature rules.

Our focus is on making the evaluation practical for a wide range of realistic parameters, corresponding to the requirements of numerical integration to an accuracy of about 100 to 100 000 bits. Our evaluation algorithm combines the summation by rectangular splitting of several types of expansions in terms of hypergeometric series with a fixed-point implementation of Bonnet's three-term recurrence relation. We then compute rigorous enclosures of the Gauss-Legendre nodes and weights using the interval Newton method. We provide rigorous error bounds for all steps of the algorithm.

The practicality of the approach is validated by an implementation in the Arb library. Our implementation achieves an order-of-magnitude speedup over previous code for computing Gauss-Legendre nodes with simultaneous high degree and precision, making Gauss-Legendre quadrature viable even at very high precision.

References

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*Inria Bordeaux, équipe LFANT, fredrik.johansson@gmail.com

†CNRS, LIP6, équipe Pequan, marc@mezzarobba.net

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