Correcting errors in a matrix inverse
(or product)

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Correcting Errors

General Framework

- Some problem is slow to solve locally.
- So we outsource the computation and get back the result.
- If the result is wrong, we should detect that.
- If the result is "close", we should correct it.

Sources of error

- Noisy communication channel
- Faults in computation
- Malicious servers?

Example: Sudoku

Sent to server

	7		8		2			
								7
1	4	8	3	5				
		5			8	3		
	6							9
3						7		
5						1	7	
		4		9				
		7	4					2

Response

9	7	6	8	1	2	4	5	3
2	5	3	9	6	4	8	1	7
1	4	5	3	8	7	9	2	6
7	9	8	6	2	5	3	4	1
4	6	1	5	7	3	2	8	9
3	8	2	1	4	9	7	6	5
5	3	9	2	9	6	1	7	4
6	2	4	7	9	1	5	3	8
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Source: `https://www.nytimes.com/crosswords/game/sudoku/hard`

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First Problem: Matrix Multiplication

Matrix Multiplication with Errors

Input: $A, B, C \in \mathsf{F}^{n\times n}$, where $C \approx AB$

Output: $E \in \mathsf{F}^{n\times n}$ s.t. $AB = C - E$

Parameters:

- Dimension n
- Input size $t = \#A + \#B + \#C$
- Error count $k = \#E$

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- Gasieniec, Levcopoulos, Lingas, Pagh, Tokuyama, *Algorithmica* 2017: $\widetilde{O}\left(n^2 + kn\right)$

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- Ours: $\widetilde{O}(t + kn)$ worst-case; $\widetilde{O}\left(t + k^{0.686}n\right)$ best case

Related Work

Monte Carlo **verification** in Linear Algebra

- Matrix multiplication: Freivalds ’79
- Positive semidefiniteness: Kaltofen, Nehring, Saunders ’11
- Rank, characteristic polynomial,. . . : Dumas & Kaltofen ’14
- Rank profile, triangular forms: Dumas, Lucas, Pernet ’17

Error correction in symbolic computation

Complexity of sparse matrix multiplication

Related Work

Monte Carlo **verification** in Linear Algebra

Error correction in symbolic computation

- Chinese remaindering: Goldreich, Ron, Sudan ’99
- Rational reconstruction: Khonji, Pernet, Roch, Roche, Stalinski ’10
- Sparse interpolation: Comer, Kaltofen, Pernet ’12

Complexity of sparse matrix multiplication

Related Work

Monte Carlo **verification** in Linear Algebra

Error correction in symbolic computation

Complexity of sparse matrix multiplication

k = number of nonzeros

- $\widetilde{O}\left(n^2 + k^{0.7}n^{1.2}\right)$: Yuster & Zwick ’04
- $\widetilde{O}\left(k^{1.34}\right)$ output sensitive: Amossen & Pagh ’09
- $\widetilde{O}\left(k^{0.188}n^2\right)$: Lingas ’10
- $\widetilde{O}\left(n^2 + kn\right)$ output sensitive: Pagh ’13

Outline

- Introduction
- Matrix Multiplication with Errors
  Crucial tools: Nonzero row identification, Sparse interpolation
- Matrix Inverse with Errors
  Additional tools: Commutativity, Sparse low-rank linear algebra
- Open Problems

Correcting errors in matrix multiplication

Input: $A, B, C \in \mathsf{F}^{n\times n}$
Output: $E \in \mathsf{F}^{n\times n}$ s.t. $AB = C - E$

Algorithm overview:

1. Formulate as $E = C - AB$
2. Find nonzero rows
   Choose random $v \in \mathsf{F}^n$ and compute Ev
3. Evaluate row polynomials
   For each evaluation θ, compute $E[1, \theta, \ldots, \theta^{n-1}]^T$
4. Interpolate row polynomials
   Sparse interpolation to recover at least half of the nonero rows
5. Repeat $O(\log k)$ times until all rows are found

Finding nonzero rows

For unknown $M \in \mathsf{F}^{n \times n}$, which rows are nonzero?

Example

M

Finding nonzero rows

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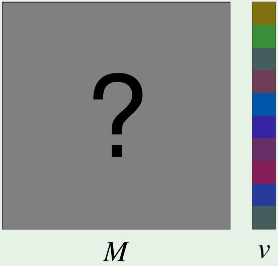
Example



1. Choose random vector $v \in \mathsf{F}^n$

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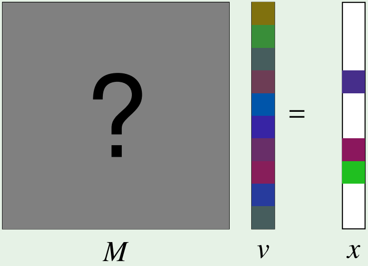

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2. Apply $Mv = x$

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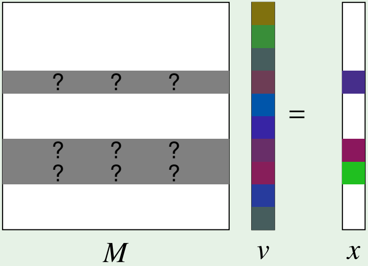
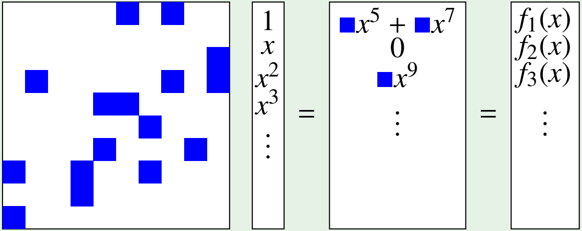


1. Choose random vector $v \in \mathsf{F}^n$
2. Apply $Mv = x$
3. Nonzero rows of M are nonzero entries in x w.h.p.

Row Polynomials

Treat each row of $M \in \mathsf{F}^{n \times n}$ as a degree-$(n-1)$ polynomial

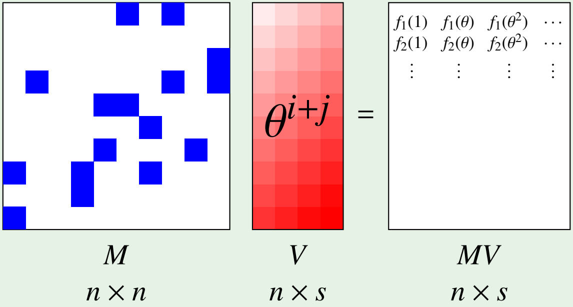
Example



Row Polynomial Evaluation

- Need to evaluate each row polynomial $f_i(x)$ at $1, \theta, \theta^2, \ldots$
- This is equivalent to multiplying by a (truncated) DFT matrix.

Example



Important: Takes $\widetilde{O}(sn + t)$ field ops, where $t = \#M$

Oligonomial Polynomial Interpolation

Algorithm (Kaltofen, Lakshman, Wiley 1990)

Unknown polynomial $f \in \mathsf{F}[x]$ with t nonzero terms
is uniquely determined by $f(1), f(\theta), f(\theta^2), \ldots, f(\theta^{2s-1})$ iff:

- f has at most s nonzero terms; and
- $\mathrm{order}(\theta) \geq \deg f$

Problem over GF(q):

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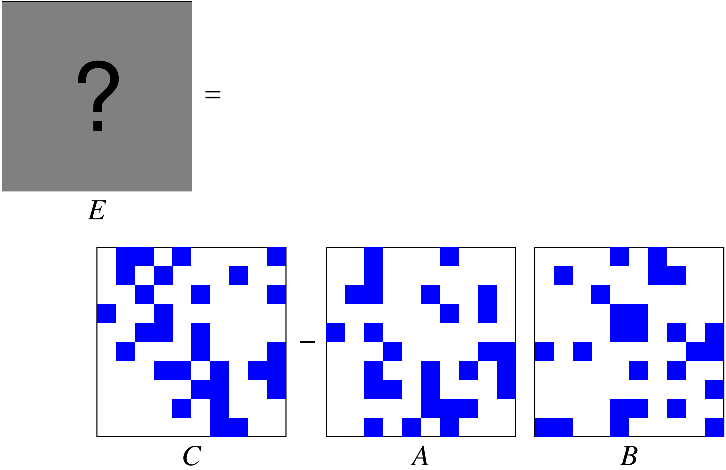
Problem over GF(q):
Needs t discrete logarithms to discover the exponents

Solution: Pre-compute first $\deg f = n$ powers of θ

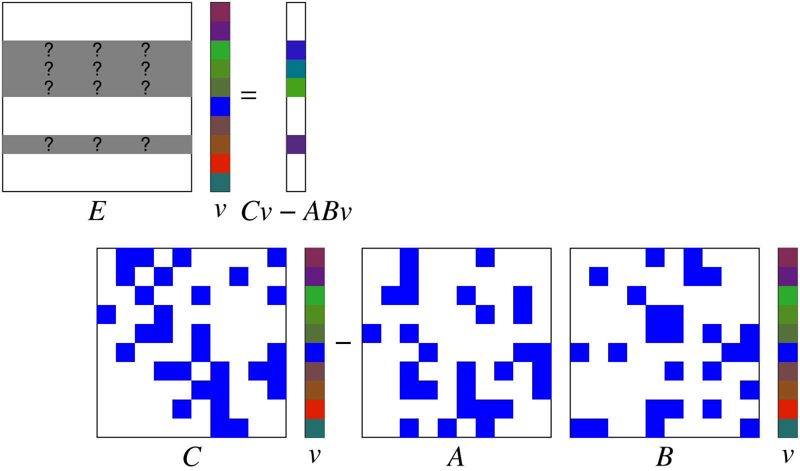
Multiplication Algorithm Overview

Given A, B, C, we want to compute $E = C - AB$



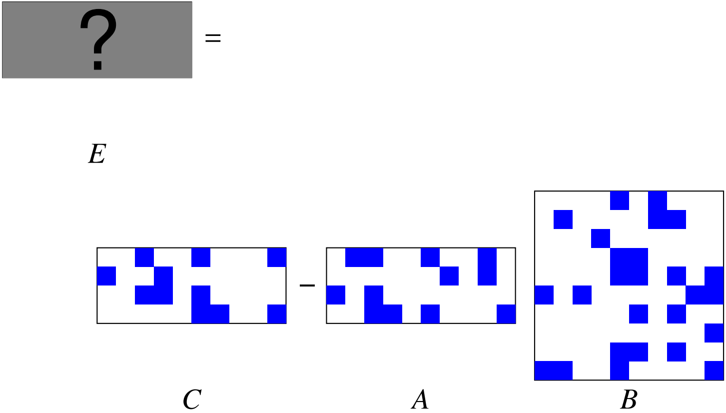
Multiplication Algorithm Overview

1. Find zero rows of E



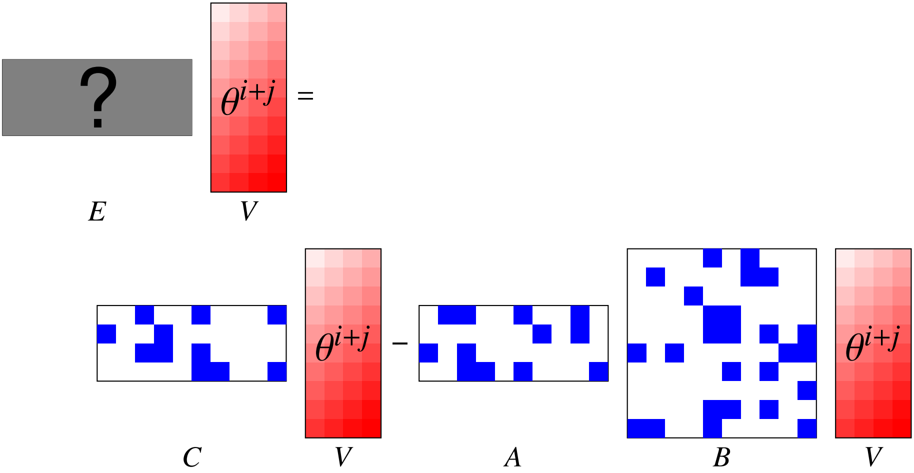
Multiplication Algorithm Overview

2 Remove corresponding rows from C and A



Multiplication Algorithm Overview

3 Choose $s \leq n$ and evaluate row polynomials at $2s$ powers of θ



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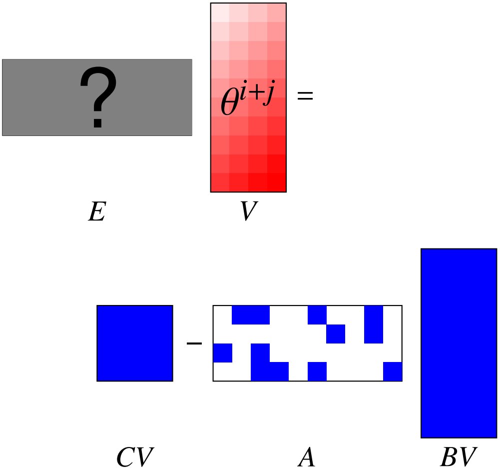

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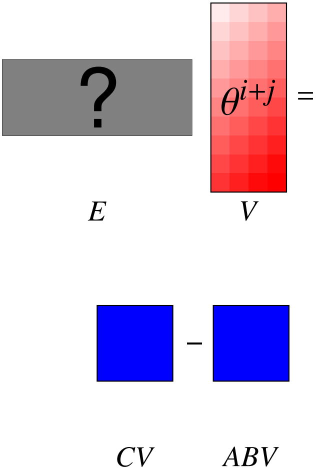

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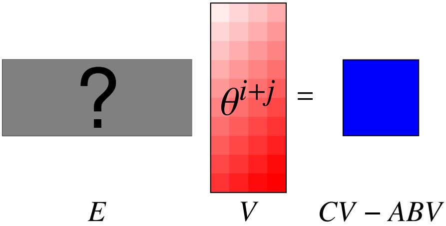

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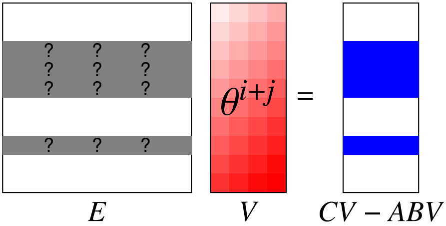
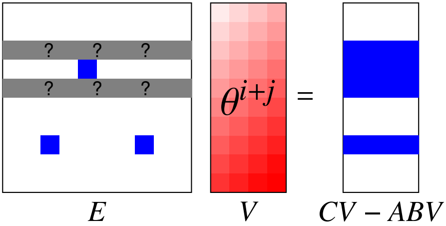


Multiplication Algorithm Overview

4. Perform sparse interpolation to recover s-sparse rows of E



Complexity Analysis

- How large should s (number of evaluations) be?
  If $s \geq 2k/r$, then we recover at least half of the nonzero rows.
- We can transpose as necessary
  Make the number of nonzero rows in E minimal.

Multiplication Complexity

Total cost in field operations is

$$\widetilde{O}\left(t + \frac{kn}{\min(r, \frac{k}{r})^{3-\omega}}\right)$$

- n = dimension
- t = number of nonzeros in A, B, C
- k = number of errors (nonzeros in E)
- r = number of nonzero rows or columns in E, whichever is smaller
- $\omega < 2.373$, exponent of matrix multiplication

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Very sparse	"Compact" errors	Worst case	
$t \ll nk^{\omega/2}$	$r \approx \sqrt{k}$	$k \le n$	$k \ge n$
$tk^{0.5}$	$t + k^{0.69}n$ $t + k^{(\omega-1)/2}n$	$t + kn$	$t + k^{0.38}n^{1.63}$ $t + k^{\omega-2}n^{4-\omega}$

Second Problem: Matrix Inverse

Matrix Inverse with Errors

Input: $A, B \in \mathsf{F}^{n \times n}$

Output: $E \in \mathsf{F}^{n \times n}$ s.t. $A^{-1} = B + E$

Parameters:

- Dimension n
- Input size $t = \#A + \#B$
- Error count $k = \#E$

- Naïve recomputation: $O(n^\omega)$
- Ours: $\widetilde{O}(t + kn + k^\omega)$ worst-case; $\widetilde{O}\left(t + k^{0.69} n\right)$ best case

Problem Formulation

Rewriting the problem definition:

$$\begin{aligned} EA &= I - BA \\ AE &= I - AB \end{aligned}$$

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$$EA = I - BA$$
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Crucial steps in multiplication algorithm:

- Computing nonzero rows of E
  Evaluate $EAv = v - A(Bv)$ for random vector v

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- Evaluating row polynomials at $2s$ powers of θ
  (Coming up next. . .)

Low-Rank Linear Algebra

Algorithm (Chung, Kwok, Lau '13)

Input: Matrix M with rank r and t nonzeros
Output: Indices of r linearly independent rows
Cost: $\widetilde{O}(t + r^{\omega})$

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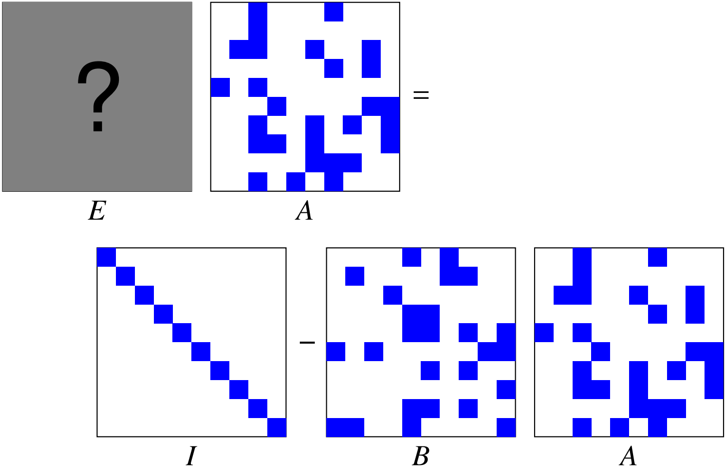
How we use this:

1. Find zero rows of E
2. Remove corresponding **columns** of A
3. Resulting $(r \times n)$ matrix has rank r
4. Remove linearly dependent rows
5. Compute inverse of resulting $r \times r$ matrix

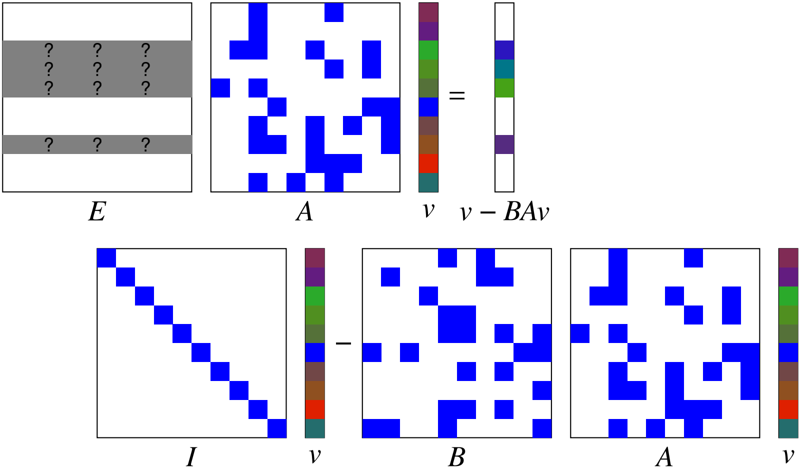
Inverse Algorithm Overview

Write $EA = I - BA$



Inverse Algorithm Overview

Find nonzero rows of E



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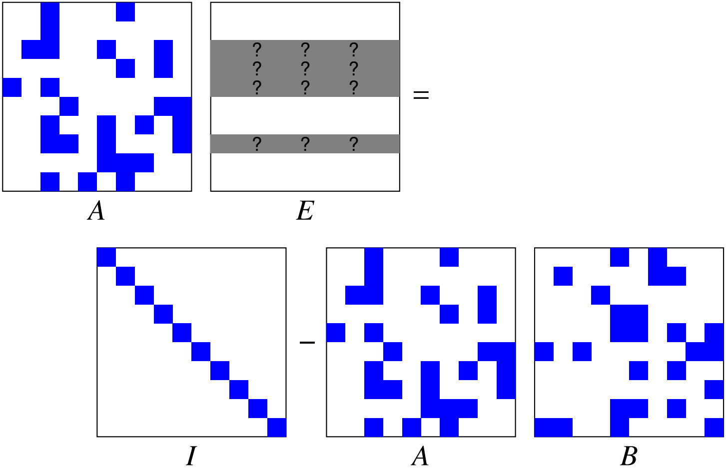
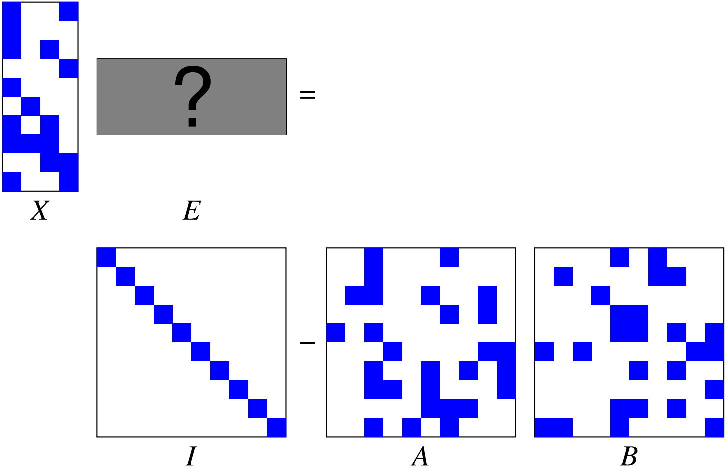


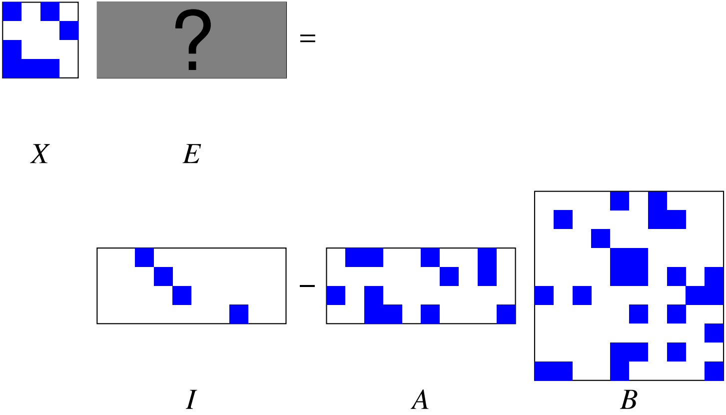
Inverse Algorithm Overview

Remove zero rows of E and corresp. columns from A



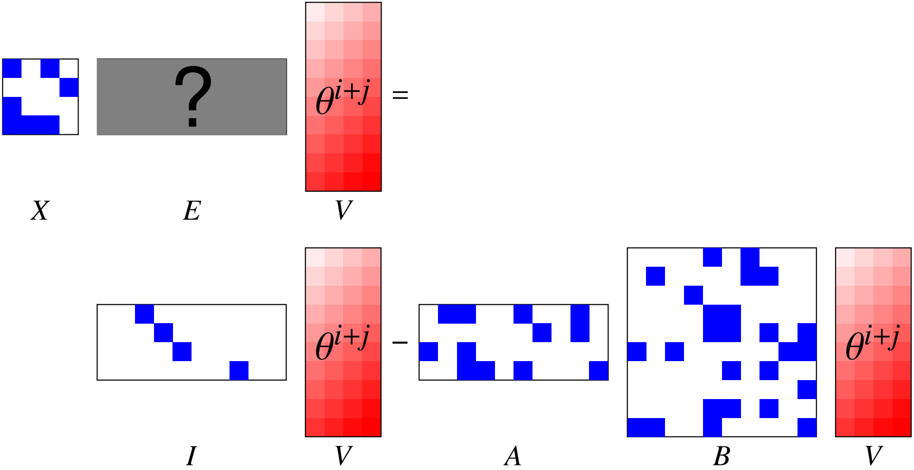
Inverse Algorithm Overview

Find linear independent rows of X, remove others from both sides



Inverse Algorithm Overview

Evaluate row polynomials at first $2s$ powers of θ



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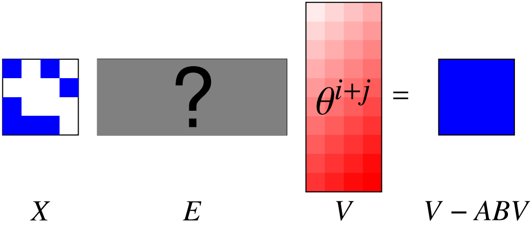
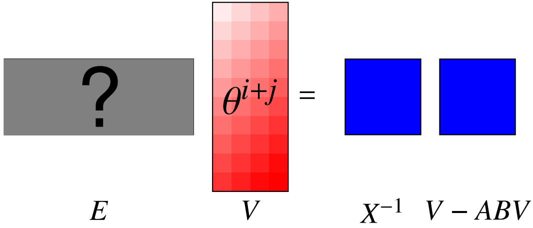


Inverse Algorithm Overview

Compute X^{-1} and apply to both sides



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(Potential) Applications

- Fault-tolerant computing
- Sparse and output-sensitive computation
- Simultaneous Chinese remaindering with different sizes

Challenge: LU Factorization

Problem

Input: $A, L, U \in \mathsf{F}^{n \times n}$
Output: $E_L, E_U \in \mathsf{F}^{n \times n}$ s.t. $A = (L + E_L)(U + E_U)$
(where L, E_L are lower triangular and U, E_U are upper triangular)

We can apply the inverse with errors algorithm except that:

- Only L **OR** U can have errors
- No commutativity, so can't choose min(nonzero rows, nonzero columns)

Next Steps

- **Faster**
  Can you achieve the “best-case” cost w.h.p.?
- **Further**
  LU factorization and other interesting problems. . .
  (Matrix inverse is kind of bullshit.)
- **Code**
  What is the *practical* number of errors k
  before recomputation is faster?


Merci !