Méthodes formelles pour les équations aux dérivées partielles

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Desiderata

Given a system of differential equations, we would like to be able to

- determine all analytic solutions;
- obtain an overview of all consequences of the system;

  in particular, given another differential equation, decide whether it is a consequence of the system or not;
- among the consequences find the ones which involve only certain specified unknowns.

Theorem (Cauchy-Kovalevskaya, 1875)

The Cauchy problem

$$\begin{cases} \dfrac{\partial u_1}{\partial z_1} & = \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{1,j,k}(z_2,\ldots,z_n,u_1,\ldots,u_m)\,\frac{\partial u_k}{\partial z_j} + b_1(z_2,\ldots,z_n,u_1,\ldots,u_m), \\ \quad\vdots & \\ \dfrac{\partial u_m}{\partial z_1} & = \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{m,j,k}(z_2,\ldots,z_n,u_1,\ldots,u_m)\,\frac{\partial u_k}{\partial z_j} + b_m(z_2,\ldots,z_n,u_1,\ldots,u_m), \\ u_1(0,z_2,\ldots,z_n) & = 0 \qquad \text{for all } z_2,\ldots,z_n, \\ \quad\vdots & \\ u_m(0,z_2,\ldots,z_n) & = 0 \qquad \text{for all } z_2,\ldots,z_n, \end{cases}$$

where $a_{i,j,k}$ and b_i are real analytic functions around the origin of $\mathbb{R}^{m+n-1}$, has a unique real analytic solution $(u_1,...,u_m)$ in a neighborhood of $(z_1,...,z_n)=(0,...,0)$.

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Systems of PDEs

A *differential system* S is given by

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

where p_1, ..., p_s and q_1, ..., q_t are polynomials in u_1, ..., u_m of z_1, ..., z_n and their partial derivatives.

Ω open and connected subset of $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$

The *solution set* of S on Ω is

$$\begin{aligned}\mathrm{Sol}_\Omega(S) \;:=\; \{\, f = (f_1, \ldots, f_m) \mid f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \, k = 1, \ldots, m, \\ p_i(f) = 0, \, q_j(f) \neq 0, \, i = 1, \ldots, s, \, j = 1, \ldots, t \,\}.\end{aligned}$$

Appropriate choice of Ω is possible only *after* formal treatment.

Systems of linear PDEs

For some l, m, $n \in \mathbb{N}$,
some ring D of differential operators,
some matrix of operators $R \in D^{l \times m}$ and
some left D-module $\mathcal{F}$ we can write

$$R\,u \;=\; 0\,, \qquad \text{where} \quad u \;=\; \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \tag{1}$$

for the unknown functions $u_i = u_i(z_1, \ldots, z_n) \in \mathcal{F}$, $i = 1, \ldots, m$.

Consequences of (1): the left D-linear combinations of the rows of R,

i.e., the elements of $D^{1 \times l}\,R$.

Linear PDEs

Example of a system of linear PDEs with constant coefficients:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} &= 0\,, \\ \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} &= 0\,, \end{cases}$$

where $u = u(x, y)$ depends on $x = z_1$ and $y = z_2$.

Choose $D = K[\partial_x, \partial_y]$, where $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$ and where ∂_x and ∂_y are the partial differential operators with respect to x and y, respectively.

The multiplication in D is composition of operators.

Linear PDEs

Example of a system of linear PDEs with non-constant coefficients:

$$\left\{ \begin{array}{l} \dfrac{\partial^3 u}{\partial x\,\partial y^2} - \dfrac{\partial^3 u}{\partial y^3} - (2y+1)\,\dfrac{\partial^2 u}{\partial y^2} - 4\,\dfrac{\partial u}{\partial y} \;=\; 0\,, \\ \dfrac{\partial^3 u}{\partial x^2\,\partial y} - \dfrac{\partial^3 u}{\partial y^3} - 2\,(2y+1)\,\dfrac{\partial^2 u}{\partial x\,\partial y} + (4y^2+4y-5)\,\dfrac{\partial u}{\partial y} \;=\; 0\,. \end{array} \right.$$

Choose K to be $\mathbb{Q}(x,y)$ or the field or meromorphic functions on some open and connected subset Ω of $\mathbb{C}^2$.

Moreover, we let $D = K\langle \partial_x, \partial_y \rangle$ be the ring of differential operators

$$\sum_{i,j \geq 0} a_{i,j}\, \partial_x^i\, \partial_y^j\,, \qquad a_{i,j} \in K\,,$$

(skew) polynomials in ∂_x and ∂_y, with non-comm. composition.

Linear PDEs

Linearizing the system on nonlinear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial x} - u^2 & = & 0\,, \\ \dfrac{\partial^2 u}{\partial y^2} - u^3 & = & 0\,, \end{array} \right. \tag{2}$$

for one unknown function u of x and y, we obtain

$$\left\{ \begin{array}{rcl} \dfrac{\partial U}{\partial x} - 2\,u\,U & = & 0\,, \\ \dfrac{\partial^2 U}{\partial y^2} - 3\,u^2\,U & = & 0\,, \end{array} \right. \tag{3}$$

for one unknown function U of x and y, where u is a solution of (2).

Preparatory treatment of the nonlinear system (2) is necessary to deal with the linearized system (3).

Linear PDEs

Thomas decomposition $\rightsquigarrow$ splitting system (2) into

$\underline{u_x} - u^2 = 0 \quad \{ \partial_x, \partial_y \}$ $2\,\underline{u_y}^2 - u^4 = 0 \quad \{ *, \partial_y \}$ $u \neq 0$	$u = 0 \quad \{ \partial_x, \partial_y \}$

Define $D = \mathbb{Q}(\sqrt{2})[u, u_x, u_y, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ and the ideal I of D which consists of all D-linear combinations of

$$u_x - u^2\,, \qquad \partial_x\left(u_x - u^2\right), \qquad \partial_y\left(u_x - u^2\right), \qquad \partial_x^2\left(u_x - u^2\right), \qquad \ldots$$

$$u_y - \tfrac{\sqrt{2}}{2}\,u^2\,, \quad \partial_x\left(u_y - \tfrac{\sqrt{2}}{2}\,u^2\right), \quad \partial_y\left(u_y - \tfrac{\sqrt{2}}{2}\,u^2\right), \quad \partial_x^2\left(u_y - \tfrac{\sqrt{2}}{2}\,u^2\right), \quad \ldots$$

$\Rightarrow$ D/I integral domain, define $K = \mathrm{Quot}(D/I)$, $D = K\langle \partial_x, \partial_y \rangle$.

(Instead of $u_y - \frac{\sqrt{2}}{2}\,u^2$ one may also choose $u_y + \frac{\sqrt{2}}{2}\,u^2$.)

1. Janet's algorithm

Janet's algorithm for linear PDEs

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{cases}$$

find: $u = u(x, y)$ analytic

$$u(x,y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \tfrac{x^2}{2!} + a_{1,1}\, \tfrac{xy}{1!1!} + a_{0,2}\, \tfrac{y^2}{2!} + \ldots$$

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Janet's algorithm for linear PDEs

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find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm for linear PDEs

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find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

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Janet's algorithm for linear PDEs

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{cases}$$

find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \dots$$

Janet's algorithm computes a vector space basis for power series solutions

(Maurice Janet, $\sim$ 1920)

Janet's algorithm for linear PDEs

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \qquad A \\ u_{x,x} - y u_{z,z} & = & 0 & \qquad B \end{array}\right.$$

is equivalent to

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \qquad A \\ u_{x,x} - y u_{z,z} & = & 0 & \qquad B \\ u_{y,z,z} & = & 0 & \qquad \tfrac{1}{2}(\partial_x^2 - y\partial_z^2)A - \tfrac{1}{2}\partial_y^2 B \\ u_{x,y,y} & = & 0 & \qquad \partial_x A \\ u_{z,z,z,z} & = & 0 & \qquad \tfrac{1}{2}(\partial_x^4 - 2y\partial_x^2\partial_z^2 + y^2\partial_z^4)A - \tfrac{1}{2}(\partial_x^2\partial_y^2 - y\partial_y^2\partial_z^2 + 2\partial_y\partial_z^2)B \\ u_{x,y,z,z} & = & 0 & \qquad \tfrac{1}{2}(\partial_x^3 - y\partial_x\partial_z^2)A - \tfrac{1}{2}\partial_x\partial_y^2 B \\ u_{x,z,z,z,z} & = & 0 & \qquad \tfrac{1}{2}(\partial_x^5 - 2y\partial_x^3\partial_z^2 + y^2\partial_x\partial_z^4)A - \tfrac{1}{2}(\partial_x^3\partial_y^2 + y\partial_x\partial_y^2\partial_z^2 - 2\partial_x\partial_y\partial_z^2)B \end{array}\right.$$

Taylor coeff's for $1,\ z,\ y,\ x,\ z^2,\ yz,\ xz,\ xy,\ z^3,\ xz^2,\ xyz,\ xz^3$ *arbitrary*,

all other coeff's determined by linear equations

Strategy

$D = K[\partial_1, \ldots, \partial_n]$, $\quad K$ field

I ideal of D (or, more generally: M submodule of D^m)

goal: compute "good" generating set for I (Janet basis)

sort terms in a polynomial in a way compatible with multiplication

total ordering $>$ on $\mathbb{M} := \mathrm{Mon}(\partial_1, \ldots, \partial_n) := \{\partial^i \mid i \in (\mathbb{Z}_{\geq 0})^n\}$

no infinitely descending chains of monomials

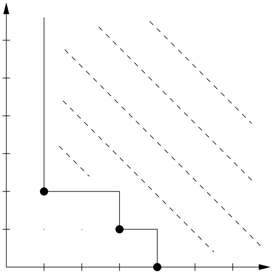
use highest terms $\mathrm{lm}(p)$, $\mathrm{lm}(q)$ to decide "divisibility"

Multiple closed sets of monomials

$$\mathbb{M} := \mathrm{Mon}(\partial_1, \ldots, \partial_n) := \{\partial^i \mid i \in (\mathbb{Z}_{\geq 0})^n\}$$

$S \subseteq \mathbb{M}$ is $\mathbb{M}$-*multiple closed* if

$$m\, s \in S \quad \forall\, m \in \mathbb{M},\ s \in S$$



$\mathbb{M}$-multiple closed set

generated by $\partial_1\partial_2^2$, $\partial_1^3\partial_2$, ∂_1^4

$=: \langle\, \partial_1\partial_2^2,\, \partial_1^3\partial_2,\, \partial_1^4\, \rangle_{\mathbb{M}}$

Multiple closed sets of monomials

Lemma

Every $\mathbb{M}$-multiple closed set $S \subseteq \mathbb{M}$ has a finite generating set.

Proof.

Every seq. $F : m_1, m_2, m_3 \ldots \in \mathbb{M}$ s.t. $m_i \not| \, m_j \quad \forall\, i < j$ is finite.

Induction: $n = 1$: clear.

$n - 1 \to n$: Let $m_1 = \partial_1^{a_1} \cdots \partial_n^{a_n}$.

Define subsequence $F^{(j,d)}$: $m_i = \partial_1^{b_1} \cdots \partial_j^{d} \cdots \partial_n^{b_n}$

We have: $$\bigcup_{1 \le j \le n} \bigcup_{0 \le d \le a_j} \{F^{(j,d)}\} = \{F\}$$

By induction, the $\{F^{(j,d)}\}$ are finite.

Multiple closed sets of monomials

Lemma

Every $\mathbb{M}$-multiple closed set $S \subseteq \mathbb{M}$ has a finite generating set.

Cor.

Every ascending sequence of $\mathbb{M}$-multiple closed sets becomes stationary.

Given a finite generating set $\{p_1, \ldots, p_r\}$ for $I \trianglelefteq K[\partial_1, \ldots, \partial_n]$,

Janet's algorithm computes

$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k = \mathrm{lm}(I)$ (all $\mathbb{M}$-multiple closed)

where S_0 is generated by $\mathrm{lm}(p_1)$, ..., $\mathrm{lm}(p_r)$

$\Rightarrow$ termination

Decomposition into disjoint cones

Def.

Let $C \subseteq \mathrm{Mon}(\partial_1, \ldots, \partial_n)$, $\mu \subseteq \{\partial_1, \ldots, \partial_n\}$.

(C, μ) is a *cone* if $\exists\, v \in C$ s.t. $C = \mathrm{Mon}(\mu)\, v$

Variables in μ: *multiplicative variables* (for v)

Variables in $\{\partial_1, \ldots, \partial_n\} - \mu$: *non-multiplicative variables* (for v)

Dimension of (C, μ): $|\mu|$

Decomposition into disjoint cones

Def.

Let $S \subseteq \mathbb{M} = \mathrm{Mon}(\partial_1, \ldots, \partial_n)$.

$\{\,(C_1, \mu_1), \ldots, (C_r, \mu_r)\,\} \subset \mathcal{P}(\mathbb{M}) \times \mathcal{P}(\{\partial_1, \ldots, \partial_n\})$

is a *decomposition of S into disjoint cones* if

each (C_i, μ_i) is a cone and $S = \dot{\bigcup}_{i=1}^{r} C_i$.

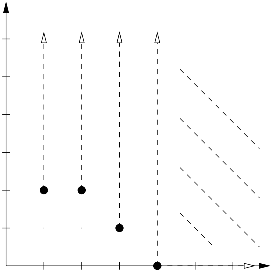
$\{\,(v_1, \mu_1), \ldots, (v_r, \mu_r)\,\}$ is a *decomp. of S into disj. cones*

if $\{\,(\mathrm{Mon}(\mu_1)\, v_1, \mu_1), \ldots, (\mathrm{Mon}(\mu_r)\, v_r, \mu_r)\,\}$ is one.

Decomposition into disjoint cones

Strategy of Janet's algorithm:

Decompose $\mathbb{M}$-multiple closed sets S into disjoint cones.



$$S = \langle\, \partial_1\partial_2^2,\, \partial_1^3\partial_2,\, \partial_1^4 \,\rangle_{\mathbb{M}}$$

decomposition:

$\partial_1\partial_2^2$	$\{\ *\ ,\partial_2\ \}$
$\partial_1^2\partial_2^2$	$\{\ *\ ,\partial_2\ \}$
$\partial_1^3\partial_2$	$\{\ *\ ,\partial_2\ \}$
∂_1^4	$\{\ \partial_1 ,\partial_2\ \}$

This can also be done for $\mathrm{Mon}(\partial_1, \ldots, \partial_n) - S$.

Janet division

The possible ways of decomposing $\mathbb{M}$-multiple closed sets

into disjoint cones are studied as

involutive divisions (Gerdt, Blinkov et. al.)

Janet division:

Let $G \subset \mathbb{M} = \mathrm{Mon}(\partial_1, \ldots, \partial_n)$ be finite.

For a cone with vertex $v = \partial_1^{a_1} \cdots \partial_n^{a_n} \in G$

∂_i is a *multiplicative variable* iff

$$a_i = \max\{\, b_i \mid \partial^b \in G;\, b_j = a_j \ \forall\, j < i \,\}.$$

Janet division

For $v = \partial_1^{a_1} \cdots \partial_n^{a_n}$:

$$\partial_i \in \mu \iff a_i = \max\{\, b_i \mid \partial^b \in G;\, b_j = a_j \ \forall\, j < i \,\}.$$

Example: $G = \{\, \partial_2\, \partial_3,\ \partial_1\, \partial_2\, \partial_3,\ \partial_1^2\, \partial_2\, \partial_3,\ \partial_1^2\, \partial_2^2 \,\}$

$\partial_2\, \partial_3$	
$\partial_1\, \partial_2\, \partial_3$	
$\partial_1^2\, \partial_2\, \partial_3$	
$\partial_1^2\, \partial_2^2$	

Janet division

For $v = \partial_1^{a_1} \cdots \partial_n^{a_n}$:

$$\partial_i \in \mu \iff a_i = \max\{\, b_i \mid \partial^b \in G;\ b_j = a_j\ \forall\, j < i \,\}.$$

Example: $G = \{\, \partial_2\, \partial_3,\ \partial_1\, \partial_2\, \partial_3,\ \partial_1^2\, \partial_2\, \partial_3,\ \partial_1^2\, \partial_2^2 \,\}$

$\partial_2\,\partial_3$	$*$	∂_2	∂_3
$\partial_1\,\partial_2\,\partial_3$	$*$	∂_2	∂_3
$\partial_1^2\,\partial_2\,\partial_3$	∂_1	$*$	∂_3
$\partial_1^2\,\partial_2^2$	∂_1	∂_2	∂_3

Decomposition into disjoint cones

Decompose(G, η) $\qquad G \subset \mathrm{Mon}(\partial_1, \ldots, \partial_n), \quad \emptyset \neq \eta \subseteq \{\partial_1, \ldots, \partial_n\}$

$G \leftarrow \{g \in G \mid \nexists\, h \in G : h \mid g\}$

if $|G| \leq 1$ or $|\mu| = 1$ then

  return $\{(m, \eta) \mid m \in G\}$

else

  $y \leftarrow y_a$ with $a = \min\{i \mid 1 \leq i \leq n,\, y_i \in \eta\}$

  $d \leftarrow \max\{\deg_y(g) \mid g \in G\}$

  $G_i \leftarrow \{g \in G \mid \deg_y(g) = i\}, \qquad i = 0, \ldots, d$

  $G_i \leftarrow G_i \cup \bigcup_{j=0}^{i-1} \{y^{i-j} g \mid g \in G_j\}, \qquad i = 1, \ldots, d$

  $T_d \leftarrow \{\, (m, \zeta \cup \{y\}) \mid (m, \zeta) \in$ Decompose$(G_d, \eta - \{y\})\,\}$

  $T_i \leftarrow$ Decompose$(G_i, \eta - \{y\}), \qquad i = 0, \ldots, d-1$

  return $\bigcup_{i=0}^{d} T_i$

fi

Janet reduction

NF(p, T, $>$) $\quad p \in K[\partial_1, \ldots, \partial_n], \quad T = \{ (d_1, \mu_1), \ldots, (d_s, \mu_s) \}$

$r \leftarrow 0$

while $p \neq 0$ do

if $\exists\, (d, \mu) \in T : \mathrm{lm}(p) \in \mathrm{Mon}(\mu)\ \mathrm{lm}(d)$ then

$p \leftarrow p - \frac{\mathrm{lc}(p)}{\mathrm{lc}(d)}\, \frac{\mathrm{lm}(p)}{\mathrm{lm}(d)}\, d$

else

$r \leftarrow r + \mathrm{lc}(p)\ \mathrm{lm}(p)$

$p \leftarrow p - \mathrm{lc}(p)\ \mathrm{lm}(p)$

fi

od

return r $\qquad$ Disjoint cones $\Rightarrow$ course of algorithm is uniquely determined

Janet's algorithm

<u>JanetBasis(F, $>$)</u> $F \subseteq K[\partial_1, \ldots, \partial_n]$ finite

$G \leftarrow F$

do

$G \leftarrow$ auto-reduce G

$J \leftarrow \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ s.t. $\{ (\mathrm{lm}(p_1), \mu_1), \ldots, (\mathrm{lm}(p_r), \mu_r) \}$

Janet decomposition of $\langle \mathrm{lm}(G) \rangle_{\mathbb{M}}$

$P \leftarrow \{ \mathrm{NF}(\partial \cdot p, J) \mid (p, \mu) \in J, \, \partial \notin \mu \}$ (passivity check)

$G \leftarrow \{ p \mid (p, \mu) \in J \} \cup P$

while $P \neq \{0\}$

return J

Janet basis

Janet basis $J = \{\, (p_1, \mu_1), \, \ldots, \, (p_r, \mu_r) \,\}$ for $I = \langle F \rangle$

- Invariant of the loop:

  G (or $\{p_1, \ldots, p_r\}$) always forms a gen. set for I

- $\displaystyle\bigcup_{i=1}^{r} \mathrm{Mon}(\mu_i) p_i$ is a K-basis of I.

  Linear independence: clear.

  $p \in I$: $\quad \displaystyle p = \sum_{i=1}^{r} c_i \, p_i$

Example

Let $I := \langle\, g_1,\, g_2 \,\rangle \trianglelefteq K[x,y]$, $\qquad g_1 := x^2 - y$, $\quad g_2 := xy - y$.

Let $>$ be degrevlex, $\; x > y$.

Decomposition into disjoint cones of $\quad \langle\, \mathrm{lm}(g_1),\, \mathrm{lm}(g_2) \,\rangle$:

$\{\, (x^2,\, \{x,y\}),\, (xy, \{y\})\,\}$

$f := x \cdot g_2 = x^2 y - xy \in I$, $\qquad f = \sum_{i=1}^{2} c_i\, g_i$?

Reduction of f modulo g_1, g_2 yields: $\qquad g_3 := y^2 - y \in I$

$\{\, (g_1, \{x,y\}),\, (g_2, \{y\}),\, (g_3, \{y\})\,\}$ $\qquad$ (minimal) Janet basis for I

Linear PDEs

Linearizing the system on nonlinear PDEs

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for one unknown function u of x and y, we obtain

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Linear PDEs

Thomas decomposition $\rightsquigarrow$ splitting system (4) into

$$\begin{array}{rcll} \underline{u_x} - u^2 & = & 0 & \{\, \partial_x, \partial_y \,\} \\ 2\,\underline{u_y}^2 - u^4 & = & 0 & \{\, *\,, \partial_y \,\} \\ u & \neq & 0 & \end{array} \qquad \begin{array}{rcll} u & = & 0 & \{\, \partial_x, \partial_y \,\} \end{array}$$

Define $D = \mathbb{Q}(\sqrt{2})[u, u_x, u_y, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ and the ideal I of D which consists of all D-linear combinations of

$$u_x - u^2\,, \quad \partial_x\left(u_x - u^2\right), \quad \partial_y\left(u_x - u^2\right), \quad \partial_x^2\left(u_x - u^2\right), \quad \ldots$$

$$u_y - \tfrac{\sqrt{2}}{2}\,u^2\,, \ \partial_x\big(u_y - \tfrac{\sqrt{2}}{2}\,u^2\big), \ \partial_y\big(u_y - \tfrac{\sqrt{2}}{2}\,u^2\big), \ \partial_x^2\big(u_y - \tfrac{\sqrt{2}}{2}\,u^2\big), \ \ldots$$

$\Rightarrow$ D/I integral domain, define $K = \mathrm{Quot}(D/I)$, $D = K\langle \partial_x, \partial_y \rangle$.

(Instead of $u_y - \frac{\sqrt{2}}{2}\,u^2$ one may also choose $u_y + \frac{\sqrt{2}}{2}\,u^2$.)

Example

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 U}{\partial y^2} - 3\,u^2\,U\right) = \frac{\partial^3 U}{\partial x\,\partial y^2} - 3\left(2\,u\,\frac{\partial u}{\partial x}\,U + u^2\,\frac{\partial U}{\partial x}\right)$$
$$= \frac{\partial^3 U}{\partial x\,\partial y^2} - 6\,u^3\,U - 6\,u^3\,U\,.$$

and

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial U}{\partial x} - 2\,u\,U\right) = \frac{\partial^3 U}{\partial x\,\partial y^2} - 2\left(\frac{\partial^2 u}{\partial y^2}\,U + 2\,\frac{\partial u}{\partial y}\,\frac{\partial U}{\partial y} + u\,\frac{\partial^2 U}{\partial y^2}\right)$$
$$= \frac{\partial^3 U}{\partial x\,\partial y^2} - 2\,u^3\,U - 2\sqrt{2}\,u^2\,\frac{\partial U}{\partial y} - 6\,u^3\,U$$

Hence, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 U}{\partial y^2} - 3\,u^2\,U\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial U}{\partial x} - 2\,u\,U\right) = 2\sqrt{2}\,u^2\,\frac{\partial U}{\partial y} - 4\,u^3\,U\,,$$

which yields the consequence

$$\frac{\partial U}{\partial y} - \sqrt{2}\,u\,U = 0\,.$$

Example

Janet basis:

$$\begin{aligned} U_x - 2\,u\,U &= 0, \quad \{\,\partial_x, \partial_y\,\}, \\ U_y \mp \sqrt{2}\,u\,U &= 0, \quad \{\,*\,, \partial_y\,\}. \end{aligned}$$

Substituting

$$u(x,y) = \frac{2}{-2x \pm \sqrt{2}y + c}, \qquad c \in \mathbb{R},$$

for u in this Janet basis results in a system of linear PDEs for U whose analytic solutions are given by

$$U(x,y) = \frac{C}{(-2x \pm \sqrt{2}y + c)^2}, \qquad C \in \mathbb{R}.$$

(Generalized) Hilbert series

Janet basis $J = \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for I

We have $\mathrm{lm}(I) = \langle \mathrm{lm}(p_1), \ldots, \mathrm{lm}(p_r) \rangle_{\mathbb{M}}$.

Generalized Hilbert series

$$H_I(\partial_1, \ldots, \partial_n) = \sum_{i=1}^{r} \mathrm{lm}(p_i) \prod_{\partial_j \in \mu_i} \frac{1}{1 - \partial_j}$$

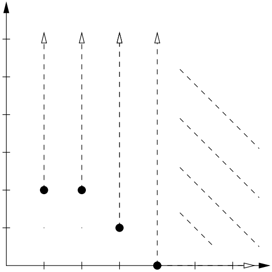
enumerates a K-basis of $\langle \mathrm{lm}(I) \rangle$.

$H_I(t, \ldots, t)$ is the usual Hilbert series.

Example

Janet basis for I

$$J = \{ (\partial_1\partial_2^2, \{\partial_2\}), (\partial_1^2\partial_2^2, \{\partial_2\}), (\partial_1^3\partial_2, \{\partial_2\}), (\partial_1^4, \{\partial_1, \partial_2\}) \}$$



generalized Hilbert series:

$$\frac{\partial_1\partial_2^2}{1-\partial_2} + \frac{\partial_1^2\partial_2^2}{1-\partial_2} + \frac{\partial_1^3\partial_2}{1-\partial_2} + \frac{\partial_1^4}{(1-\partial_1)(1-\partial_2)}$$

Hilbert polynomial

Janet basis of M: $\{\, (p_1, \mu_1), \ldots, (p_r, \mu_r) \,\}$

$$
\begin{aligned}
H_M(t, \ldots, t) &= \sum_{k \geq 0} \dim_K M_k \, t^k \\
&= \sum_{i=1}^{r} t^{\deg(p_i)} \frac{1}{(1-t)^{|\mu_i|}} \\
&= \sum_{i=1}^{r} t^{\deg(p_i)} \sum_{j \geq 0} \binom{|\mu_i| + j - 1}{j} t^j
\end{aligned}
$$

Coeff. of t^k in $H_M(t, \ldots, t)$? For $k \geq \max\{\deg(p_i) \mid i = 1, \ldots, r\}$:

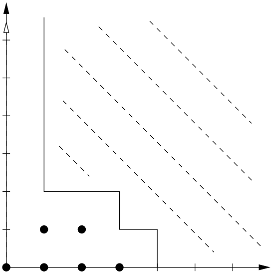
$$
\dim_K M_k = \sum_{i=1}^{r} \binom{|\mu_i| + k - \deg(p_i) - 1}{k - \deg(p_i)}
$$

Example

$S = \mathrm{lm}(I)$

Decomposition of $\mathrm{Mon}(\partial_1, \ldots, \partial_n) - S$ into disjoint cones

$\rightsquigarrow$ generalized Hilbert series enum. a K-basis of $K[\partial_1, \ldots, \partial_n]/I$



generalized Hilbert series:

$$\frac{1}{1-\partial_2} + \partial_1 + \partial_1\partial_2 + \partial_1^2 + \partial_1^2\partial_2 + \partial_1^3$$

Hilbert polynomial:

$$\sum_{i=1}^{6} \binom{|\mu_i| + k - \deg(p_i) - 1}{k - \deg(p_i)} = 1$$

Power series solutions

$$\frac{\partial^2 u}{\partial x\, \partial y} = 0, \quad \{\ *\ , \partial_y\, , \partial_z\ \},$$
$$\frac{\partial^3 u}{\partial x^2\, \partial y} = 0, \quad \{\ *\ , \partial_y\, , \partial_z\ \},$$
$$\frac{\partial^4 u}{\partial x^3\, \partial z} = 0, \quad \{\ \partial_x\, ,\ *\ , \partial_z\ \},$$
$$\frac{\partial^4 u}{\partial x^3\, \partial y} = 0, \quad \{\ \partial_x\, , \partial_y\, , \partial_z\ \}.$$

Janet decomposition of the set of parametric derivatives / generalized Hilbert series:

$$\begin{array}{rl} 1, & \{\ *\ , \partial_y\, , \partial_z\ \}, \\ \partial_x, & \{\ *\ ,\ *\ , \partial_z\ \}, \\ \partial_x^2, & \{\ *\ ,\ *\ , \partial_z\ \}, \\ \partial_x^3, & \{\ \partial_x\, ,\ *\ ,\ *\ \}. \end{array} \qquad \frac{1}{(1-\partial_y)(1-\partial_z)} + \frac{\partial_x}{1-\partial_z} + \frac{\partial_x^2}{1-\partial_z} + \frac{\partial_x^3}{1-\partial_x}.$$

Accordingly, a formal power series solution u is uniquely determined as

$$u(x,y,z) = f_0(y,z) + x\, f_1(z) + x^2\, f_2(z) + x^3\, f_3(x)$$

by any choice of formal power series f_0, f_1, f_2, f_3 of the indicated variables.

Power series solutions

$$\left\{ \begin{array}{lcl} \dfrac{\partial^2 u}{\partial x^2} & = & f(x,y) \\ \dfrac{\partial^2 u}{\partial y^2} & = & g(x,y) \end{array} \right.$$

complete:

$$\left\{ \begin{array}{rcll} \dfrac{\partial^2 u}{\partial x^2} & = & f(x,y) & \{\ \partial_x\,,\,\partial_y\ \}, \\ \dfrac{\partial^2 u}{\partial y^2} & = & g(x,y) & \{\ *\ ,\,\partial_y\ \}, \\ \dfrac{\partial^3 u}{\partial x \partial y^2} & = & \dfrac{\partial g}{\partial x} & \{\ *\ ,\,\partial_y\ \} \end{array} \right.$$

integrability condition:

$$\left. \begin{array}{l} \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 g}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial^2 u}{\partial x^2}\right) = \dfrac{\partial^2 f}{\partial y^2} \end{array} \right\} \quad \Longrightarrow \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 g}{\partial x^2}\,.$$

Power series solutions

Janet decomposition of the set of parametric derivatives:

$$
\begin{array}{rl}
1, & \{\ *\ ,\ *\ \}, \\
\partial_x, & \{\ *\ ,\ *\ \}, \\
\partial_y, & \{\ *\ ,\ *\ \}, \\
\partial_x \partial_y, & \{\ *\ ,\ *\ \}.
\end{array}
$$

initial conditions:

$$
\left\{
\begin{array}{rcl}
u(0,0) & = & a_{0,0}, \\
\dfrac{\partial u}{\partial x}(0,0) & = & a_{1,0}, \\
\dfrac{\partial u}{\partial y}(0,0) & = & a_{0,1}, \\
\dfrac{\partial^2 u}{\partial x \partial y}(0,0) & = & a_{1,1}
\end{array}
\right.
$$

$$
u(x,y) = a_{0,0} + a_{0,1}\, y + a_{1,0}\, x + a_{1,1}\, x\, y + \sum_{i \geq 2,\, j \geq 0} a_{i,j}\, \frac{x^i}{i!}\, \frac{y^j}{j!} + \sum_{j \geq 2} a_{0,j}\, \frac{y^j}{j!} + \sum_{j \geq 2} a_{1,j}\, x\, \frac{y^j}{j!}
$$

Power series solutions

Janet decomposition of the set of parametric derivatives:

$$\begin{array}{rl} 1, & \{\ *\ ,\ *\ \}, \\ \partial_x, & \{\ *\ ,\ *\ \}, \\ \partial_y, & \{\ *\ ,\ *\ \}, \\ \partial_x\partial_y, & \{\ *\ ,\ *\ \}. \end{array}$$

initial conditions:

$$\left\{\begin{array}{rcl} u(0,0) & = & a_{0,0}, \\ \dfrac{\partial u}{\partial x}(0,0) & = & a_{1,0}, \\ \dfrac{\partial u}{\partial y}(0,0) & = & a_{0,1}, \\ \dfrac{\partial^2 u}{\partial x \partial y}(0,0) & = & a_{1,1} \end{array}\right.$$

$$u(x,y) = a_{0,0} + \ldots + a_{1,1}\, x\, y + \int_0^x\int_0^x f(x,y)\,\mathrm{d}x\,\mathrm{d}x + \int_0^y\int_0^y g(0,y)\,\mathrm{d}y\,\mathrm{d}y + x\int_0^y\int_0^y \frac{\partial g}{\partial x}(0,y)\,\mathrm{d}y\,\mathrm{d}y$$

Desiderata

Given a system of differential equations, we would like to be able to

- determine all analytic solutions;
- obtain an overview of all consequences of the system;

  in particular, given another differential equation, decide whether it is a consequence of the system or not;
- among the consequences find the ones which involve only certain specified unknowns.

Elimination

Lemma

$J \subseteq R := K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ Janet basis w.r.t. any term order.

For any $0 \neq p \in R$ let $\mathrm{lm}(p)$ be its leading monomial.

If $\{\, p \in J \mid p \in K[Y_1, \ldots, Y_m] \,\} = \{\, p \in J \mid \mathrm{lm}(p) \in K[Y_1, \ldots, Y_m] \,\}$,

then $J \cap K[Y_1, \ldots, Y_m]$ generates $\langle J \rangle \cap K[Y_1, \ldots, Y_m]$.

Proof. Let $0 \neq p \in \langle J \rangle \cap K[Y_1, \ldots, Y_m]$. Since J is a Janet basis,

$\exists\, q \in J$, $\mathrm{lm}(q) \in K[Y_1, \ldots, Y_m]$, $\mathrm{lm}(q) \mid \mathrm{lm}(p)$.

By assumption, $q \in K[Y_1, \ldots, Y_m]$.

Reduction $p \to 0$ in $K[Y_1, \ldots, Y_m]$. $\square$

Free resolution

Janet basis $J = \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for I

We have $\partial_j \, p_i = \sum_k \alpha_{i,j,k} \, p_k,$ $\quad \partial_j \notin \mu_i, \quad \alpha_{i,j,k} \in K[\mu_k]$

Define $\pi : D^{|J|} \to D : \hat{p}_i \mapsto p_i$. $\quad$ ($\hat{p}_i$ std. gen.)

Prop.

$$\partial_j \, \hat{p}_i - \sum_k \alpha_{i,j,k} \, \hat{p}_k, \qquad \partial_j \notin \mu_i, \quad i = 1, \ldots, r,$$

form a Janet basis of $\ker \pi$ for a suitable monomial ordering.

$\rightsquigarrow$ construction of a free resolution of D/I.

Example

$$D = K[\partial_1, \partial_2, \partial_3], \qquad \partial_1 > \partial_2 > \partial_3, \qquad I = (\partial_1, \partial_2, \partial_3)$$

Janet basis:

$$\begin{array}{c|ccc} \partial_1 & \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & * & \partial_2 & \partial_3 \\ \partial_3 & * & * & \partial_3 \end{array}$$

normal form computation:

$$\begin{array}{l} \partial_1 \cdot \partial_2 - \partial_2 \cdot \partial_1 \\ \partial_1 \cdot \partial_3 - \partial_3 \cdot \partial_1 \\ \partial_2 \cdot \partial_3 - \partial_3 \cdot \partial_2 \end{array}$$

$$D^{1\times 3} \xrightarrow{\begin{pmatrix} -\partial_2 & \partial_1 & 0 \\ -\partial_3 & 0 & \partial_1 \\ 0 & -\partial_3 & \partial_2 \end{pmatrix}} D^{1\times 3} \xrightarrow{\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}} D \longrightarrow D/I \longrightarrow 0$$

Example

$$D = K[\partial_1, \partial_2, \partial_3], \qquad \partial_1 > \partial_2 > \partial_3, \qquad I = (\partial_1, \partial_2, \partial_3)$$

Janet basis:

$$\begin{array}{lll|ccc}
[-\partial_2 & \underline{\partial_1} & 0] & \partial_1 & \partial_2 & \partial_3 \\
[-\partial_3 & 0 & \underline{\partial_1}] & \partial_1 & \partial_2 & \partial_3 \\
[\quad 0 & -\partial_3 & \underline{\partial_2}] & * & \partial_2 & \partial_3
\end{array}$$

normal form computation:

$$\partial_1 \cdot [\quad 0 \;\; -\partial_3 \quad \partial_2] - \partial_2 \cdot [-\partial_3 \quad 0 \quad \partial_1] + \partial_3 \cdot [-\partial_2 \quad \partial_1 \quad 0]$$

$$0 \to D \xrightarrow{\begin{pmatrix} \partial_3 & -\partial_2 & \partial_1 \end{pmatrix}} D^{1\times 3} \xrightarrow{\begin{pmatrix} -\partial_2 & \partial_1 & 0 \\ -\partial_3 & 0 & \partial_1 \\ 0 & -\partial_3 & \partial_2 \end{pmatrix}} D^{1\times 3} \xrightarrow{\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}} D \to D/I \to 0$$

Free resolution

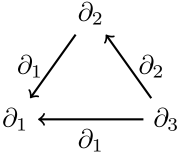
Prop.

$$\partial_j \, \hat{p}_i - \sum_k \alpha_{i,j,k} \, \hat{p}_k, \qquad \partial_j \notin \mu_i, \quad i = 1, \ldots, r,$$

form a Janet basis of $\ker \pi$ w.r.t. $\prec$.

Choose a total order $\ll$ on J s.t.

$p_k \ll p_l$ if $\exists$ path from p_l to p_k in the Janet graph.



$$\partial^i \, \hat{p}_k \prec \partial^j \, \hat{p}_l \quad :\Longleftrightarrow \quad \begin{cases} \partial^i \; \mathrm{lm}(p_k) < \partial^j \; \mathrm{lm}(p_l) \\ \text{or} \quad \partial^i \; \mathrm{lm}(p_k) = \partial^j \; \mathrm{lm}(p_l) \quad \text{and} \quad p_k \ll p_l \end{cases}$$

Consequences

$\{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ Janet basis for $I \trianglelefteq R$

- can decide ideal membership
- normal form for residue classes modulo I
- enumeration of a K-basis of I and a K-basis of R/I (generalized Hilbert series)
- can easily determine Hilbert polynomial
- can read off a free resolution of R/I
- every Janet basis is a Gröbner basis

Janet bases over $\mathbb{Z}$

NF(p, T, $\prec$) $\qquad p \in \mathbb{Z}[x_1, \ldots, x_n], \quad T = \{ (d_1, \mu_1), \ldots, (d_l, \mu_l) \}$

$r \leftarrow 0$

while $p \neq 0$ do

  if $\exists (d, \mu) \in T : \mathrm{lm}(p) \in \mathrm{Mon}(\mu)\, d$ then

    write $\mathrm{lc}(p) = a \cdot \mathrm{lc}(d) + b, \qquad |b| < |\,\mathrm{lc}(d)|$

    if $a \neq 0$ then

      $p \leftarrow p - a\, \frac{\mathrm{lm}(p)}{\mathrm{lm}(d)}\, d$

    else

      move leading term from p *to* r

    fi

  else

    move leading term from p *to* r

  fi

od

return r

Janet bases for Ore algebras

Skew polynomial ring $A[\partial; \sigma, \delta]$:

A domain and K-algebra

$\sigma : A \to A$ K-algebra endomorphism

$\delta : A \to A$ σ-derivation, i.e.

$$\delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b, \qquad a, b \in A$$

$$A[\partial; \sigma, \delta] = \left\{ \sum_{\text{fin.}} a_i\,\partial^i \mid a_i \in A,\ i \in \mathbb{Z}_{\geq 0} \right\}$$

with commutation rule

$$\partial\,a = \sigma(a)\,\partial + \delta(a), \qquad a \in A$$

Janet bases for Ore algebras

Ore algebra $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$:

$A = K$ or $A = K[x_1, \ldots, x_n]$

$\sigma_i : D \to D$ K-algebra endomorphisms

$\delta_i : D \to D$ σ_i-derivations

$$A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m] = \left\{ \sum_{\text{fin.}} a_i \, \partial^i \mid a_i \in A, \, i \in (\mathbb{Z}_{\geq 0})^m \right\}$$

with commutation rules

$$\begin{aligned} \partial_i \, a &= \sigma_i(a) \, \partial_i + \delta_i(a), \qquad a \in A, \\ \partial_i \, \partial_j &= \partial_j \, \partial_i \end{aligned}$$

Janet bases for Ore algebras

- Weyl algebra: *ordinary differential equations*

  $A_1 = K[t][\frac{d}{dt}]$

  $$\boxed{\frac{d}{dt}\, a = a\, \frac{d}{dt} + \frac{da}{dt}}$$

- Weyl algebra: *partial differential equations*

  $A_n = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$

  $$\boxed{\partial_i\, x_j = x_j\, \partial_i + \delta_{ij}}$$

- $B_n = K(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$

- Shift operators: *difference equations*

  $S_h = K[t][\delta_h]$

  $$\boxed{\delta_h\, t = (t - h)\, \delta_h}$$

- combinations ...

Janet bases for Ore algebras

$D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_m]$

I *left* ideal of D generated by p_1, ..., p_r

- *normal form* for elements of D:

  use $\partial_i \, x_j = \sigma_i(x_j) \, \partial_i + \ldots$

  to move all ∂_i to the right of every x_j

- $\mathbb{M} := \{ \, x^i \, \partial^j \mid i \in (\mathbb{Z}_{\geq 0})^n, \, j \in (\mathbb{Z}_{\geq 0})^m \, \}$

  consider $\mathbb{M}$-multiple closed set generated by

  the normal forms of $\mathrm{lm}(p_i)$, $i = 1, \ldots, r$

- decomp. into disj. cones as before

- reduction: all multiplications from the *left*

Janet bases for Ore algebras

$D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_m]$

I left ideal of D generated by p_1, ..., p_r

For *termination* of the algorithm, assume that

$$\partial_i \, x_j = (c_{i,j} \, x_j + d_{i,j}) \, \partial_i + e_{i,j}$$

where $c_{i,j} \in K - \{0\}$, $d_{i,j} \in K$,

$e_{i,j} \in K[x_1, \ldots, x_n]$ with $\deg(e_{i,j}) \leq 1$

Involutive

- Janet (-like Gröbner) bases for submodules of free modules over a commutative polynomial ring
- coefficients: rationals or finite fields and field extensions, and rational integers
- Janet division, Janet-like division
- term orderings:
  degrevlex, plex
  TOP / POT
  block / elimination orderings

web: `http://wwwb.math.rwth-aachen.de/Janet`

Involutive

- Analogues of Buchberger's criteria can be selected
- Interface to C++:
  call fast routines when needed or
  switch to fast routines for the whole Maple session
- Syzygies, Hilbert series, etc.
- Applications:
  commutative algebra
  solving systems of algebraic equations

web: `http://wwwb.math.rwth-aachen.de/Janet`

Main procedures of Involutive

`InvolutiveBasis`

compute Janet(-like Gröbner) basis

`PolInvReduce`

involutive reduction modulo Janet basis

`FactorModuleBasis`

vector space basis of residue class module

`Syzygies`

syzygy module

`PolResolution`

free resolution

`PolHilbertSeries`, `PolHilbertPolynomial`, etc.

combinatorial devices

`PolMinPoly`, `PolRepres`, etc.

computing in residue class rings

Janet

- Janet (-like Gröbner) bases for linear systems of partial differential equations
- Janet division, Janet-like division
- analogues of Buchberger's criteria can be selected
- computational tools for differential operators
- elementary divisor algorithm for $K(x)[\partial]$ (Jacobson normal form)
- parametric derivatives
- formal power series solutions, polynomial solutions

web: `http://wwwb.math.rwth-aachen.de/Janet`

Main procedures of Janet

`JanetBasis`

compute Janet(-like Gröbner) basis

`InvReduce`

involutive reduction modulo Janet basis

`ParamDeriv`

parametric derivatives

`CompCond`, `Resolution`

compatibility conditions (syzygies)

`HilbertSeries`, `HilbertPolynomial`, etc.

combinatorial devices

`SolSeries`, `PolySol`

formal power series / polynomial solutions

`ElementaryDivisors`

Jacobson normal form

ginv

- C++ module for Python
- comp. of Gröbner bases using involutive algorithms
- polynomials, differential / difference equations
- open source software
- originated by V. P. Gerdt, Y. A. Blinkov
- contributions by LBfM
- coefficients: rationals or finite fields and some algebraic and transcendental field extensions
- term orderings: degrevlex (TOP / POT), lex, product orderings
- see web page for timings

web: `http://invo.jinr.ru`

ginv

```
import ginv
st = ginv.SystemType("Polynomial")
im = ginv.MonomInterface("DegRevLex", st, ['x', 'y'])
ic = ginv.CoeffInterface("GmpZ", st)
ip = ginv.PolyInterface("PolyList", st, im, ic)
iw = ginv.WrapInterface("CritPartially", ip)
iD = ginv.DivisionInterface("Janet", iw)
eqs = ["x^2+y^2", ...]
basis = ginv.basisBuild("TQ", iD, eqs)
```

Involutive Basis Algorithm (Gerdt)

choose $f \in F$ with the lowest $\mathrm{lm}(f)$ w.r.t. $\prec$

$G \leftarrow \{f\}$; $Q \leftarrow F - G$

do

  $h \leftarrow 0$

  while $Q \neq \emptyset$ and $h = 0$ do

    choose $p \in Q$ with the lowest $\mathrm{lm}(p)$ w.r.t. $\prec$

    $Q \leftarrow Q - \{p\}$; $h \leftarrow \mathrm{NF}(p, G, \prec)$

  if $h \neq 0$ then

    for all $\{g \in G \mid \mathrm{lm}(g) = x^i\, \mathrm{lm}(h),\ |i| > 0\}$ do

      $Q \leftarrow Q \cup \{g\}$; $G \leftarrow G - \{g\}$

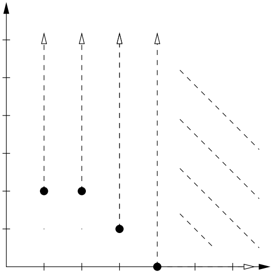
    $G \leftarrow G \cup \{h\}$

    $Q \leftarrow Q \cup \{\, x \cdot g \mid g \in G,\ x$ non-mult. for $g\,\}$

while $Q \neq \emptyset$

return G

Janet-like Gröbner Bases



Idea: do not store all

the *prolongations*

$\rightsquigarrow$ Janet-like division

Note: In general, the minimal Gröbner basis is still a proper subset of the Janet-like Gröbner basis.

V. P. Gerdt, Y. A. Blinkov, *Janet-like Monomial Division. Janet-like Gröbner Bases.* CASC 2005, LNCS 3781, Springer, 2005

Formal Power Series

Let $D = K[z_1, ..., z_n][\partial_1, ..., \partial_l]$ s.t. Janet bases can be computed.

Theorem

$\mathcal{F} := \hom_K(D, K)$ is an injective left D-module.

The pairing $(\ ,\ ) : D \times \mathcal{F} \to K : (d, \lambda) \mapsto \lambda(d)$ is non-degenerate:

$\lambda \in \mathcal{F}$ is uniquely determined by $\lambda(d)$, $d \in D$

$\lambda \in \mathcal{F}$ is uniquely determined by $\lambda(m)$, $m \in \mathrm{Mon}(D)$

Formal Power Series

Thm. $\mathcal{F} := \hom_K(D, K)$ is an injective left D-module.

Proof. By Baer's criterion, consider w.l.o.g. left ideal I of D and $\varphi : I \to \mathcal{F}$. Extension $\widehat{\varphi} : D \to \mathcal{F}$?

System of D-linear equations $d \cdot \lambda = \varphi(d)$, $d \in I$, $\lambda \in \mathcal{F}$.

J Janet basis for I $\Rightarrow$ $\bigcup_{i=1}^{r} \mathrm{Mon}(\mu_i) p_i$ is a K-basis of I

choose values $(m, \lambda) = \lambda(m)$ for $m \in \mathrm{Mon}(D) - \mathrm{lm}(I)$

$\Rightarrow$ values $(\mathrm{lm}(p), \lambda)$ uniquely determined $\Rightarrow$ solution $\lambda \in \mathcal{F}$

$\widehat{\varphi}$ defined by $\widehat{\varphi}(1) := \lambda$.

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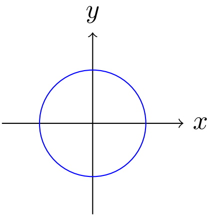
Algebraic Geometry

$$\begin{cases} \mathbb{R} \to \mathbb{R}^2 \\ t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \end{cases}$$



$$x^2 + y^2 - 1 = 0$$

Eliminate t in $\quad x = \dfrac{2t}{t^2+1}, \quad y = \dfrac{t^2-1}{t^2+1} \quad \dots$

Special Solutions

$$\begin{aligned} \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \frac{1}{\rho} \nabla p &= 0 \qquad \text{(Navier-Stokes)} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0 \end{aligned}$$

cylindrical coordinates r, θ, z, $\rho \equiv 1$ (incompressible flow)

Ansatz: $v_i(r, \theta, z) = f_i(r) g_i(\theta) h_i(z)$, $i = 1,\ 2,\ 3$

PDE: $u u_{x,y} - u_x u_y = 0$, $u \in \{v_1, v_2, v_3\}$, $(x, y) \in \{(r, \theta), (r, z), (\theta, z)\}$

one of the many simple systems of the Thomas decomposition:

$$\begin{aligned} v(t, r, \theta, z) &= \left(-\frac{(t+c_2) F_1(t)}{r} - \frac{r}{2(t+c_2)},\ \frac{(\theta+c_1) r}{t+c_2},\ 0 \right), \\ p(t, r, \theta, z) &= (t + c_2) \ln(r) \dot{F}_1(t) - \frac{(t+c_2)^2 F_1(t)^2}{2r^2} + (\ln(r) + (\theta + c_1)^2) F_1(t) \\ &\quad + F_2(t) - \frac{2\nu \ln(r)}{t+c_2} + \frac{((\theta+c_1)^2 - \frac{3}{4}) r^2}{2(t+c_2)^2}. \end{aligned}$$

2. Thomas decomposition of differential systems

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Systems of PDEs

A *differential system* S is given by

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

where p_1, ..., p_s and q_1, ..., q_t are polynomials in u_1, ..., u_m of z_1, ..., z_n and their partial derivatives.

Ω open and connected subset of $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$

The *solution set* of S on Ω is

$$\begin{aligned}\mathrm{Sol}_\Omega(S) \; := \; \{\, f = (f_1, \ldots, f_m) \mid f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \, k = 1, \ldots, m, \\ p_i(f) = 0, \, q_j(f) \neq 0, \, i = 1, \ldots, s, \, j = 1, \ldots, t \,\}.\end{aligned}$$

Appropriate choice of Ω is possible only *after* formal treatment.

Systems of PDEs

A *differential system* S is given by

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

Consequences of the system obtained in a finite number of steps from:

- $p_1 = 0$, $p_2 = 0$, ..., $p_s = 0$ are consequences,
- if $p = 0$ is consequence, then any partial derivative of $p = 0$ is,
- if $p \cdot q = 0$ is consequence and q a factor of some q_i, then $p = 0$ is consequence,
- if $p = 0$, $r = 0$ are consequences, then $a\,p + b\,r = 0$ is (a, b differential polynomials)

Polynomial ODEs / PDEs

$$\left(\frac{du}{dt}\right)^2 - 4\,t\,\frac{du}{dt} - 4\,u + 8\,t^2 = 0$$

find: $u = u(t)$ analytic

$$u(t) = a_0 + a_1\,t + a_2\,\tfrac{t^2}{2!} + a_3\,\tfrac{t^3}{3!} + \ldots$$

Substitute and compare coefficients:

$$\begin{cases} a_1^2 - 4\,a_0 = 0 & \qquad a_0 := 0 \quad \Rightarrow \quad a_1 = 0 \\ 2\,a_1\,a_2 - 8\,a_1 = 0 & \\ a_1\,a_3 + a_2^2 - 6\,a_2 + 8 = 0 & \qquad \Rightarrow \quad (a_2 - 2)(a_2 - 4) = 0 \\ \vdots & \end{cases}$$

Many case distinctions?

Thomas' algorithm $\rightsquigarrow$ finitely many so-called simple systems

(Joseph Miller Thomas, $\sim$ 1930)

Algebraic geometry

$$L \;=\; \{\; p_1(x_1, ..., x_n) = 0,\; ...,\; p_r = 0,\; q_1 \neq 0,\; ...,\; q_s \neq 0 \;\}$$

polynomial equations (and inequations)

$$\mathrm{Sol}(L) \;=\; \{\; a \in \mathbb{C}^n \mid p_i(a) = 0,\; q_j(a) \neq 0 \;\forall\, i, j \;\}$$

Conversely, let $S \subseteq \mathbb{C}^n$.

$$\mathcal{I}(S) \;=\; \{\; p \in \mathbb{C}[x_1, ..., x_n] \mid p(a) = 0 \;\forall\, a \in S \;\}$$

Nullstellensatz (Hilbert, 1893) (for equations)

radical ideals of $\mathbb{C}[x_1, ..., x_n]$ $\longleftrightarrow$ zero sets in $\mathbb{C}^n$

are bijections which are inverse to each other.

Differential algebraic geometry

Differential algebra (Ritt, Kolchin, Seidenberg, ...)

$\mathbb{Q} \subseteq K$ a differential field with commuting derivations ∂_1, ..., ∂_n

Differential polynomial ring with derivations ∂_1, ..., ∂_n

$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$

$K\{u\}$ not Noetherian (e.g., $[u'u''$, $u''u''', \ldots] \subseteq K\{u\}$ not fin. gen.)

Thm. (Ritt-Raudenbush).
Every radical diff. ideal of $K\{u_1, \ldots, u_m\}$ is finitely generated, is intersection of finitely many prime diff. ideals.

Thm. (Differential Nullstellensatz).
Every radical diff. ideal $I \subsetneq K\{u_1, \ldots, u_m\}$ has a zero in a diff. field ext. of K. If $f \in K\{u_1, \ldots, u_m\}$ vanishes for all zeros of I, then $f \in I$.

Thomas Decomposition

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

algebraic reduction:

$$p = u_{x,x,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$p \;\rightarrow\; r = c \cdot p - u_{x,x,y} \cdot q$$

differential reduction:

$$p = u_{x,x,y,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$\partial_y\, q = \frac{\partial q}{\partial u_{x,x,y}}\, u_{x,x,y,y} + \ldots$$

$$p \;\rightarrow\; r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

reduction requires: initial $c \neq 0$ and separant $\frac{\partial q}{\partial u_{x,x,y}} \neq 0$

Thomas Decomposition

$R = K\{u_1, \ldots, u_m\}$

Def. *Thomas decomposition* of diff. system S (or $\mathsf{Sol}(S)$):

$\mathsf{Sol}(S) = \mathsf{Sol}(S_1) \uplus \ldots \uplus \mathsf{Sol}(S_r)$, S_i simple diff. system

Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E diff. ideal generated by p_1, ..., p_s

q product of initials and separants of all p_i

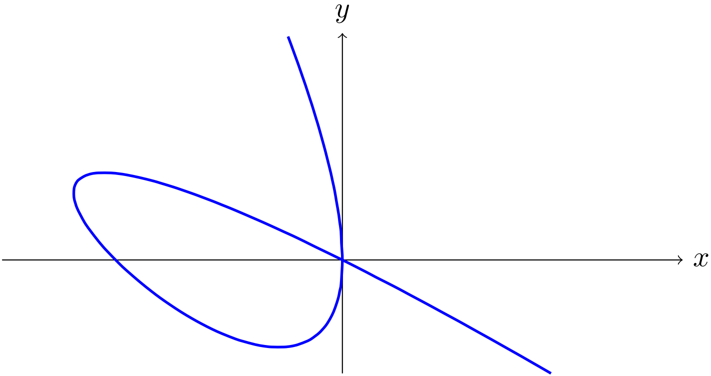
Then

$$E : q^\infty := \{ p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \} = \mathcal{I}_R(\mathsf{Sol}(S))$$

consists of all diff. polynomials in R vanishing on $\mathsf{Sol}(S)$.

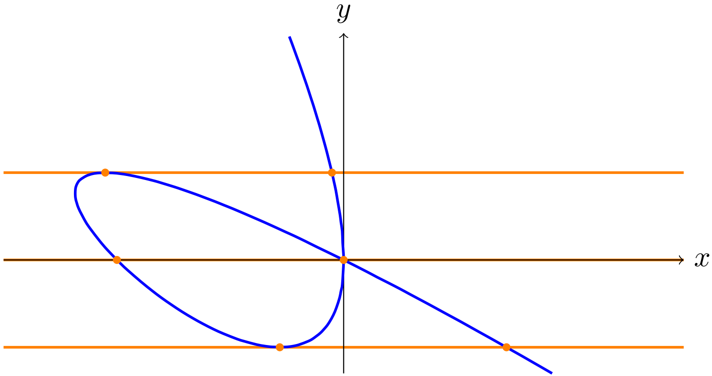
Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



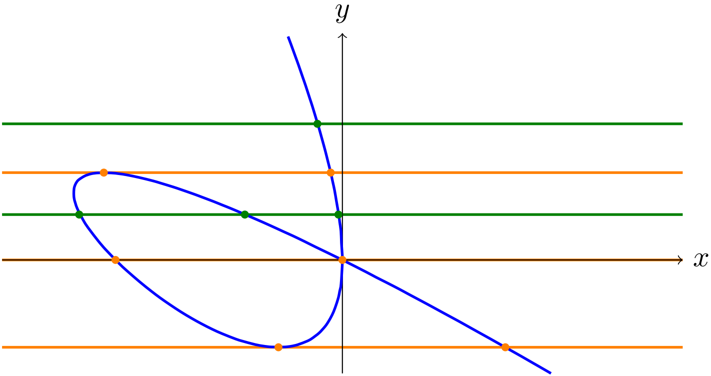
Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



Thomas Decomposition

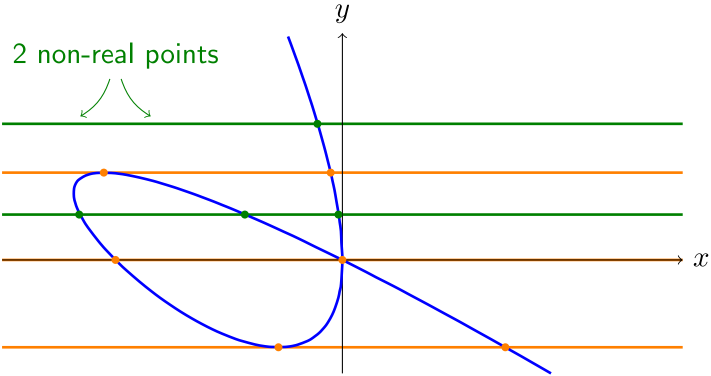
$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Simple Systems

K field of char. 0, $p_1, \ldots, p_s$, $q_1, \ldots, q_t \in K[x_1, \ldots, x_n]$

$$V = \left\{ a \in \overline{K}^n \,\middle|\, p_i(a) = 0,\ q_j(a) \neq 0 \quad \forall\, i, j \right\}$$

$$\pi_i : \overline{K}^{n-(i-1)} \longrightarrow \overline{K}^{n-i} : (a_i, a_{i+1}, \ldots, a_n) \longmapsto (a_{i+1}, \ldots, a_n)$$

$V_1 := V$, $V_{i+1} := \pi_i(V_i)$

V is *simple*, if for each i one of the following three cases holds:

$$\begin{cases} \exists\, e \quad \forall (a_{i+1}, \ldots, a_n) \in \pi_i(V_i) \quad \exists!\, a_i^{(1)}, \ldots, a_i^{(e)} \quad (a_i^{(j)}, a_{i+1}, \ldots, a_n) \in V_i, \\ \exists\, f \quad \forall (a_{i+1}, \ldots, a_n) \in \pi_i(V_i) \quad \exists!\, a_i^{(1)}, \ldots, a_i^{(f)} \quad (a_i^{(j)}, a_{i+1}, \ldots, a_n) \notin V_i, \\ \qquad \forall (a_{i+1}, \ldots, a_n) \in \pi_i(V_i) \qquad (a_i, a_{i+1}, \ldots, a_n) \in V_i \quad \forall a_i \in \overline{K} \end{cases}$$

Thomas decomposition: Write $V = W_1 \uplus \ldots \uplus W_r$ where W_j simple

Simple Systems

$$p_1, \ldots, p_s, \quad q_1, \ldots, q_t \in K[x_1, \ldots, x_n], \qquad x_1 > x_2 > \ldots > x_n$$

$$V = \left\{ a \in \overline{K}^n \,\middle|\, p_i(a) = 0,\ q_j(a) \neq 0 \quad \forall\, i, j \right\}$$

Identify $K[x_1, \ldots, x_n] = K[x_n][x_{n-1}] \ldots [x_1]$.

$S = \{p_1 = 0, \ldots, p_s = 0,\ q_1 \neq 0, \ldots, q_t \neq 0\}$ is a *simple system*, if

1. Each variable is leader of at most one p_i or q_j.
2. The initial of p_i, q_j has no zero in $\pi_k(V_k)$,
   if x_k is the leader of p_i resp. q_j.
3. $p_i(x_k, a_{k+1}, \ldots, a_n)$, $q_j(x_k, a_{k+1}, \ldots, a_n)$ are square-free
   for all $(a_{k+1}, \ldots, a_n) \in \pi_k(V_k)$,
   if x_k is the leader of p_i resp. q_j.

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$$a\underline{x}^2 + b\underline{x} + c \ = \ 0$$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c \;=\; 0$ $a \;\neq\; 0$	$b\underline{x} + c \;=\; 0$ $a \;=\; 0$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c = 0$	$2a\underline{x} + b = 0$	$b\underline{x} + c = 0$
$4a\underline{c} - b^2 \neq 0$	$4a\underline{c} - b^2 = 0$	
$a \neq 0$	$a \neq 0$	$a = 0$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c = 0$	$2a\underline{x} + b = 0$	$b\underline{x} + c = 0$	
$4a\underline{c} - b^2 \neq 0$	$4a\underline{c} - b^2 = 0$		$c = 0$
		$b \neq 0$	$b = 0$
$a \neq 0$	$a \neq 0$	$a = 0$	$a = 0$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

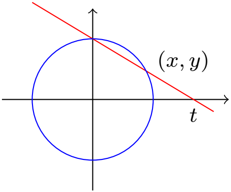
$$\boxed{\begin{array}{rcl} a\underline{x}^2 + b\underline{x} + c & = & 0 \\ 4a\underline{c} - b^2 & \neq & 0 \\ & & \\ a & \neq & 0 \end{array}} \quad \boxed{\begin{array}{rcl} 2a\underline{x} + b & = & 0 \\ 4a\underline{c} - b^2 & = & 0 \\ & & \\ a & \neq & 0 \end{array}} \quad \boxed{\begin{array}{rcl} b\underline{x} + c & = & 0 \\ & & \\ b & \neq & 0 \\ a & = & 0 \end{array}} \quad \boxed{\begin{array}{rcl} & & \\ c & = & 0 \\ b & = & 0 \\ a & = & 0 \end{array}}$$

$x_1 \neq x_2$ | $x_1 = x_2$ | x_1 | all $x \in \overline{\mathbb{Q}}$

solve $p(x) = 0$ for fixed choice of a, b, c

Thomas Decomposition

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ (1-y)\,t - x = 0 \end{cases}$$



$$p_1 := x^2 + y^2 - 1, \qquad p_2 := x + t\,y - t \quad \in \quad \mathbb{Q}[x, y, t], \qquad x > y > t$$

$$p_1 - (x - ty + t)\,p_2 \;=\; (y-1)\,((t^2+1)y - t^2 + 1)$$

Thomas decomposition:

$$\boxed{\begin{array}{rcl} (t^2+1)\,\underline{x} - 2t &=& 0 \\ (t^2+1)\,\underline{y} - t^2 + 1 &=& 0 \\ \underline{t}^2 + 1 &\neq& 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} &=& 0 \\ \underline{y} - 1 &=& 0 \end{array}}$$

Thomas Decomposition

$S = \{ p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \}$

Def. *Thomas decomposition* of diff. system S (or $\mathrm{Sol}(S)$):

$\mathrm{Sol}(S) = \mathrm{Sol}(S_1) \uplus \ldots \uplus \mathrm{Sol}(S_r)$, S_i simple diff. system

Def. S is *simple* if

(a) $p_1, \ldots, p_s, q_1, \ldots, q_t$ have pairwise distinct leaders,

(b) leading coefficients and discriminants of p_i and q_j do not vanish,

(c) $p_1, \ldots, p_s$ form a passive PDE system,

(d) $q_1, \ldots, q_t$ are reduced modulo $p_1, \ldots, p_s$.

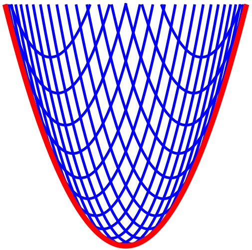
set of *admissible derivations* $\mu_i \subseteq \{\partial_1, \ldots, \partial_n\}$ for p_i, $i = 1, \ldots, s$

Thomas Decomposition

$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad\qquad p \in \mathbb{Q}(t)\{u\}$$

Separant of p: $\frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{u}}, \dot{u}) = -16u + 16t^2$



Thomas decomposition:

$p = 0$ $u - t^2 \neq 0$	$u - t^2 = 0$

general solution: $u(t) = 2((t+c)^2 + c^2), \qquad c \in \mathbb{R}$

essential singular solution: $u(t) = t^2$

Thomas Decomposition

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\ \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{array} \right.$$

Define

$$p_1 \ := \ u_{x,x} - u_{y,y}, \qquad p_2 \ := \ u_x - u^2$$

$R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y.

degree-reverse lexicographical ranking $>$ on R with $\partial_x u > \partial_y u$

$$p_3 \ := \ p_1 - \partial_x p_2 - 2\,u\,p_2 \ = \ -u_{y,y} + 2\,u^3 \,.$$

Janet division associates the sets of admissible derivations:

$$\left\{ \begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_{y,y}} - 2\,u^3 & = & 0, & \{\, *\,, \partial_y\} \end{array} \right.$$

Thomas Decomposition

passivity check:

$$\begin{aligned} \partial_x p_3 + \partial_y^2 p_2 - 6\,u^2 p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4) \\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,. \end{aligned}$$

factorization $\quad\rightsquigarrow\quad$ splitting possible

If the above factorization is ignored, then the discriminant of $p_4 := u_y^2 - u^4$ needs to be considered, which implies vanishing or non-vanishing of the separant $2\,u_y$. This case distinction leads to the

Thomas decomposition

$$\boxed{\begin{aligned} \underline{u_x} - u^2 &= 0, \quad \{\partial_x, \partial_y\} \\ \underline{u_y}^2 - u^4 &= 0, \quad \{\,*\,, \partial_y\} \\ \underline{u} &\neq 0 \end{aligned}} \qquad \boxed{\underline{u} = 0}$$

Thomas Decomposition

passivity check:

$$\begin{aligned} \partial_x p_3 + \partial_y^2 p_2 - 6\,u^2\,p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4) \\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,. \end{aligned}$$

factorization $\quad\rightsquigarrow\quad$ splitting possible

For both systems a differential reduction of p_3 modulo the chosen factor is applied because the monomial ∂_y defining the new leader divides the monomial $\partial_{y,y}$ defining $\mathrm{ld}(p_3)$. We obtain the

Thomas decomposition

$$\begin{array}{|c|} \hline \begin{aligned} \underline{u_x} - u^2 &= 0, \quad \{\partial_x, \partial_y\} \\ \underline{u_y} + u^2 &= 0, \quad \{\,*\,, \partial_y\} \end{aligned} \\ \hline \end{array} \qquad \begin{array}{|c|} \hline \begin{aligned} \underline{u_x} - u^2 &= 0, \quad \{\partial_x, \partial_y\} \\ \underline{u_y} - u^2 &= 0, \quad \{\,*\,, \partial_y\} \\ \underline{u} &\neq 0. \end{aligned} \\ \hline \end{array}$$

Thomas Decomposition

$$\frac{\partial u}{\partial t} - 6\,u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \qquad u\,\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial u}{\partial t}\,\frac{\partial u}{\partial x} = 0,$$

$R := K\{u\}$ with commuting derivations ∂_t, ∂_x

degree-reverse lexicographical ranking on R with $u_t > u_x$

Define

$$p \;:=\; u_t - 6\,u\,u_x + u_{x,x,x}, \qquad q \;:=\; u\,u_{t,x} - u_t\,u_x$$

$\mathrm{ld}(p) = u_{x,x,x}$, $\mathrm{ld}(q) = u_{t,x}$, $\mathrm{init}(p) = 1$, and $\mathrm{init}(q) = u$. Hence,

$$\{\, p = 0,\; q = 0 \,\}$$

is a triangular set. We replace this system with two systems

$$\{\, p = 0,\; q = 0,\; u = 0 \,\}, \quad \{\, p = 0,\; q = 0,\; u \neq 0 \,\}$$

according to vanishing or non-vanishing initial of q.

Thomas Decomposition

The first system is equivalent to the simple differential system

$$S_1 := \{ \, u = 0 \, \}.$$

The second system is simple as an algebraic system, but not passive.

Janet division associates the sets of admissible derivations:

$$\left\{ \begin{array}{rcll} p & := & u_t - 6\,u\,u_x + u_{x,x,x}\,, & \{\, * \,, \partial_x \} \\ q & := & u\,u_{t,x} - u_t\,u_x\,, & \{ \partial_t\,, \partial_x \} \end{array} \right.$$

Janet reduction of $\partial_t \, p$ modulo $\{ \, (p, \{\partial_x\}), \, (q, \{\partial_t, \partial_x\}) \, \}$ yields

$$\begin{array}{rcl} r & := & u\,(u\,p_t - q_{x,x}) - u\,u_t\,p + u_x\,q_x \\ & = & u^2\,\underline{u_{t,t}} - u\,(6\,u^2 - u_{x,x})\,u_{t,x} - u_t\,u_x\,u_{x,x} - u\,u_t^2. \end{array}$$

Thomas Decomposition

$\{ p = 0, \, q = 0, \, r = 0, \, u \neq 0 \}$ is simple as an algebraic system.

Janet reduction of $\partial_t \, q$ modulo $\{ (p, \{\partial_x\}), \, (q, \{\partial_x\}), \, (r, \{\partial_t, \partial_x\}) \}$:

$$\begin{aligned} s \quad := \quad & u\,((u\,q_t - r_x) - (6\,u^2 - u_{x,x})\,q_x + q\,p) \; + \\ & 3\,u_x\,r + 3\,(2\,u^2\,u_x - u\,u_t - u_x\,u_{x,x})\,q \;\; = \;\; 6\,u^3\,u_t\,\underline{u_{x,x}}. \end{aligned}$$

We have $\mathrm{init}(s) = 6\,u^3\,u_t$. Now, $\mathrm{init}(s) \neq 0$ implies $u_{x,x} = 0$, which results in the simple system

$$S_2 := \{ u_t - 6\,u\,u_x = 0, \, u_{x,x} = 0, \, u \neq 0 \}.$$

On the other hand, $\mathrm{init}(s) = 0$ implies $u_t = 0$, hence the simple system

$$S_3 := \{ u_t = 0, \, u_{x,x,x} - 6\,u\,u_x = 0, \, u_{x,x} \neq 0, \, u \neq 0 \}.$$

Thomas decomposition: $\quad S_1, \quad S_2, \quad S_3$

Thomas Decomposition

$u_t - 6uu_x + u_{x,x,x} = 0$ (Korteweg-de Vries)

Ansatz: $u(t,x) = f(t)g(x)$

PDE: $uu_{t,x} - u_t u_x = 0$

Thomas decomposition of $\{u_t - 6uu_x + \underline{u_{x,x,x}} = 0,\ u\underline{u_{t,x}} - u_t u_x = 0\}$:

$u = 0$	$u_t - 6uu_x = 0$ $u_{x,x} = 0$ $u \neq 0$	$u_t = 0$ $u_{x,x,x} - 6uu_x = 0$ $u_{x,x} \neq 0$ $u \neq 0$
solutions:	$u(t,x) = \frac{x+c_1}{-6t+c_2}$	$x = \pm \int_{u(0)}^{u(x)} \frac{dz}{\sqrt{2z^3 - az - b}}$

Thomas Decomposition

- 1937: J. M. Thomas: “Differential Systems”.
- 1998: D. Wang: implementation for algebraic systems
- 2007: V. Gerdt: “On decomposition of algebraic PDE systems into simple subsystems”
- 2009: W. Plesken: “Counting solutions of polynomial systems via iterated fibrations”
- since 2009: implementations in Maple for
  - algebraic systems (T. Bächler)
  - systems of PDEs (M. Lange-Hegermann)

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. R.,
Algorithmic Thomas decomposition of algebraic and differential systems,
J. Symbolic Computation 47(10):1233–1266, 2012.

Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

Thomas decomposition:

$$\begin{array}{rcl} \det(\ldots) & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & \neq & 0 \\ u_{z,z} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} -u_{y,z}^2 u_{x,x} + 2u_{y,z}u_{x,z}u_{x,y} - u_{y,y}u_{x,z}^2 & = & 0 \\ u_{y,z} & \neq & 0 \\ u_{z,z} & = & 0 \end{array}$$

$$\begin{array}{rcl} u_{z,z}u_{x,y} - u_{y,z}u_{x,z} & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & = & 0 \\ u_{z,z} & \neq & 0 \end{array}$$

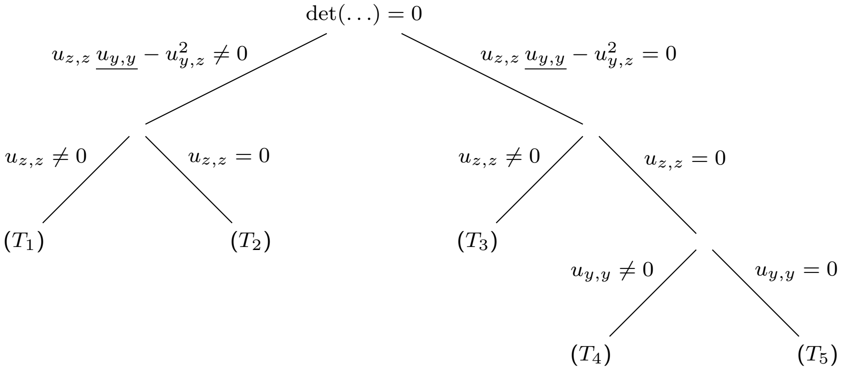
$$\begin{array}{rcl} u_{x,z} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \\ u_{y,y} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} u_{y,y} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \end{array}$$

Example

$$\det\begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking



Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

DifferentialAlgebra (Maple 17):

$$\begin{array}{rcl} \det(\ldots) & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & \neq & 0 \end{array}$$

$$\begin{array}{rcl} u_{z,z}u_{x,y} - u_{y,z}u_{x,z} & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & = & 0 \\ u_{z,z} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} u_{x,z} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \end{array}$$

$$\begin{array}{rcl} u_{y,y} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \end{array}$$

Thomas Decomposition

$R = K\{u_1, \ldots, u_m\}$

Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E diff. ideal generated by p_1, ..., p_s,

q product of initials and separants of all p_i. Then

$$E : q^\infty := \{\, p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\} = \mathcal{I}_R(\mathsf{Sol}(S))$$

consists of all diff. polynomials in R vanishing on $\mathsf{Sol}(S)$.

Thm. S not necessarily simple

S_1, ..., S_r Thomas decomposition of S w.r.t. any ranking on R

$$\sqrt{E : q^\infty} = \left(E^{(1)} : (q^{(1)})^\infty\right) \cap \ldots \cap \left(E^{(r)} : (q^{(r)})^\infty\right)$$

Elimination

Lemma

$J \subseteq R := K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ Janet basis w.r.t. any term order.

For any $0 \neq p \in R$ let $\mathrm{lm}(p)$ be its leading monomial.

If $\{ p \in J \mid p \in K[Y_1, \ldots, Y_m] \} = \{ p \in J \mid \mathrm{lm}(p) \in K[Y_1, \ldots, Y_m] \}$,

then $J \cap K[Y_1, \ldots, Y_m]$ generates $\langle J \rangle \cap K[Y_1, \ldots, Y_m]$.

Proof. Let $0 \neq p \in \langle J \rangle \cap K[Y_1, \ldots, Y_m]$. Since J is a Janet basis,

$\exists\, q \in J$, $\mathrm{lm}(q) \in K[Y_1, \ldots, Y_m]$, $\mathrm{lm}(q) \mid \mathrm{lm}(p)$.

By assumption, $q \in K[Y_1, \ldots, Y_m]$.

Reduction $p \to 0$ in $K[Y_1, \ldots, Y_m]$. $\square$

Differential Elimination

Lemma

Let S be simple, w.r.t. any ranking $>$, E diff. ideal generated by

$S^{=} = \{ p_1, \ldots, p_s \}$, q prod. init. sep. of all p_i, $V \subset \{ u_1, \ldots, u_m \}$

If $P := \{ p \in S^{=} \mid p \in K\{V\} \} = \{ p \in S^{=} \mid \mathrm{ld}(p) \in \mathrm{Mon}(\Delta)\, V \}$,

then $(E : q^\infty) \cap K\{V\} = E' : (q')^\infty$,

E' diff. ideal of $K\{V\}$ gen. by P, q' prod. of init. and sep. of $p \in P$.

Proof. Let $0 \neq p \in (E : q^\infty) \cap K\{V\}$. Since S is simple,

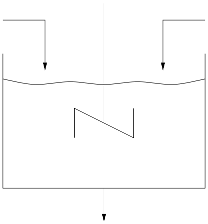
$b\, p = R$-linear comb. of $p_1, \ldots, p_s$ and their derivatives

By assumption, for every Janet divisor is in $K\{V\}$.

Pseudo-reduction $p \to 0$ in $K\{V\}$. □

Nonlinear control systems

Stirred tank:

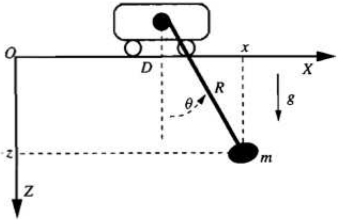


$$\begin{cases} \dot{V}(t) = F_1(t) + F_2(t) - k\sqrt{V(t)} \\ \dot{\overline{c(t)\,V(t)}} = c_1\,F_1(t) + c_2\,F_2(t) - c(t)\,k\sqrt{V(t)} \end{cases}$$

H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, 1972.

Nonlinear control systems

2-D crane:

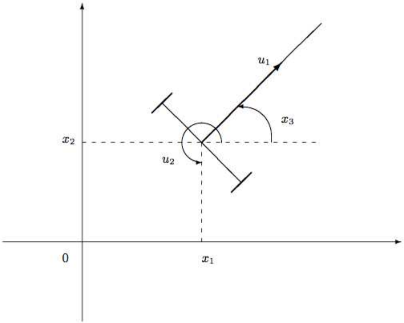


$$\begin{cases} m\,\ddot{x} &= -T\,\sin\theta \\ m\,\ddot{z} &= -T\,\cos\theta + m\,g \\ x &= R\,\sin\theta + d \\ z &= R\,\cos\theta \end{cases}$$

M. Fliess, J. Lévine, P. Martin, P. Rouchon, *Flatness and defect of non-linear systems: introductory theory and examples*, Internat. J. Control 61(6), 1327–1361, 1995.

Nonlinear control systems

Unicycle:



$$\begin{cases} \dot{x}_1 = \cos(x_3)\, u_1 \\ \dot{x}_2 = \sin(x_3)\, u_1 \\ \dot{x}_3 = u_2 \end{cases}$$

H. Nijmeijer, A. van der Schaft, *Nonlinear dynamical control systems*, Springer, 1990.

G. Conte, C. H. Moog, A. M. Perdon, *Nonlinear control systems*, Vol. 242 of LNCIS, Springer, 1999.

Algebraic approach to systems theory

A few references:

S. Diop, *Elimination in control theory*,
Math. Control Signals Systems 4(1):17–32, 1991.

S. Diop, *Differential-algebraic decision methods and some applications to system theory*, Theoret. Comput. Sci. 98(1):137–161, 1992.

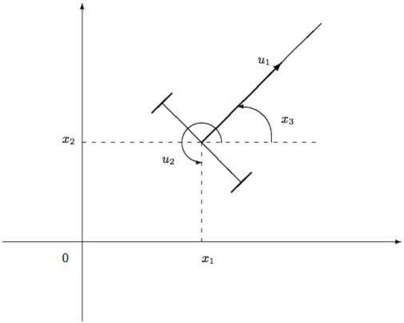
M. Fliess, S. T. Glad, *An Algebraic Approach to Linear and Nonlinear Control*, in: H. L. Trentelman and J. C. Willems (eds.), *Essays on Control: Perspectives in the Theory and its Applications*, pp. 223–267, Birkhäuser, 1993.

M. Fliess, J. Lévine, P. Martin, P. Rouchon,
Flatness and defect of non-linear systems: introductory theory and examples, Internat. J. Control 61(6):1327–1361, 1995.

J.-F. Pommaret, *Partial differential control theory*, Kluwer, 2001.

Example

Unicycle:



$$\begin{cases} \dot{x}_1 = \cos(x_3)\, u_1 \\ \dot{x}_2 = \sin(x_3)\, u_1 \\ \dot{x}_3 = u_2 \end{cases}$$

H. Nijmeijer, A. van der Schaft, *Nonlinear dynamical control systems*, Springer, 1990.

G. Conte, C. H. Moog, A. M. Perdon, *Nonlinear control systems*, Vol. 242 of LNCIS, Springer, 1999.

Example

$R = \mathbb{Q}\{x_1, x_2, cx3, sx3, u_1, u_2, y_1, y_2\}$

block ranking $>$ with $\{x_1, x_2, cx3, sx3\} \gg \{u_1, u_2\} \gg \{y_1, y_2\}$

```
> with(DifferentialThomas):

> ivar := [t]:  dvar := [x1,x2,cx3,sx3,u1,u2,y1,y2]:

> ComputeRanking(ivar,
[[x1,x2,cx3,sx3],[u1,u2],[y1,y2]]):

> L := [x1[t]-cx3*u1, x2[t]-sx3*u1, sx3[t]-cx3*u2,
y1-x1, y2-x2, cx3^2+sx3^2-1]:

> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [x1_1 - cx3_0 u1_0,\quad x2_1 - sx3_0 u1_0,\quad sx3_1 - cx3_0 u2_0,\quad y1_0 - x1_0,\quad y2_0 - x2_0,\quad {cx3_0}^2 + {sx3_0}^2 - 1]$$

```
> TD := DifferentialThomasDecomposition(LL, [cx3]);
```

$$TD := [DifferentialSystem, DifferentialSystem, DifferentialSystem, DifferentialSystem, DifferentialSystem]$$

Example

```
> Print(TD[1]);
```

$$[\underline{x1} - y1 = 0, \quad \underline{x2} - y2 = 0, \quad u1\,\underline{cx3} - y1_t = 0, \quad u1\,\underline{sx3} - y2_t = 0,$$
$$\underline{u1}^2 - {y1_t}^2 - {y2_t}^2 = 0, \quad {y1_t}^2\,\underline{u2} + {y2_t}^2\underline{u2} - y1_t y2_{t,t} + y2_t y1_{t,t} = 0,$$
$$\underline{y2_t} \neq 0, \quad \underline{y1_t} \neq 0, \quad \underline{y1_t}^2 + {y2_t}^2 \neq 0, \quad y2_t\underline{y1_{t,t}} - y1_t y2_{t,t} \neq 0]$$

```
> collect(%[6], u2, factor);
```

$$\left({y1_t}^2 + {y2_t}^2\right) u2 - y1_t y2_{t,t} + y2_t y1_{t,t} = 0$$

```
> Print(TD[2]);
```

$$[\underline{x1} - y1 = 0, \quad \underline{x2} - y2 = 0, \quad u1\,\underline{cx3} - y1_t = 0, \quad u1\,\underline{sx3} - y2_t = 0,$$
$$\underline{u1}^2 - {y1_t}^2 - {y2_t}^2 = 0, \quad \underline{u2} = 0, \quad y2_t\underline{y1_{t,t}} - y1_t y2_{t,t} = 0,$$
$$\underline{y2_t} \neq 0, \quad \underline{y1_t} \neq 0, \quad \underline{y1_t}^2 + {y2_t}^2 \neq 0]$$

$$\begin{vmatrix} (y_1)_t & (y_2)_t \\ (y_1)_{t,t} & (y_2)_{t,t} \end{vmatrix} = 0 \qquad \Rightarrow \qquad \dot{x}_1 \text{ and } \dot{x}_2 \text{ are proportional}$$

Example

```
> Print(TD[3]);
```

$$[\underline{x1} - y1 = 0, \quad \underline{x2} - y2 = 0, \quad \underline{cx3} + 1 = 0, \quad \underline{sx3} = 0, \quad \underline{u1} + y1_t = 0,$$
$$\underline{u2} = 0, \quad \underline{y2_t} = 0, \quad \underline{y1_t} \neq 0]$$

```
> Print(TD[4]);
```

$$[\underline{x1} - y1 = 0, \quad \underline{x2} - y2 = 0, \quad \underline{cx3} - 1 = 0, \quad \underline{sx3} = 0, \quad \underline{u1} - y1_t = 0,$$
$$\underline{u2} = 0, \quad \underline{y2_t} = 0, \quad \underline{y1_t} \neq 0]$$

movement restricted to any of the two directions defined by the x_1-coordinate, no rotation allowed

```
> Print(TD[5]);
```

$$[\underline{x1} - y1 = 0, \quad \underline{x2} - y2 = 0, \quad \underline{cx3}^2 + sx3^2 - 1 = 0, \quad \underline{sx3_t} - u2\, cx3 = 0,$$
$$\underline{u1} = 0, \quad \underline{y1_t} = 0, \quad \underline{y2_t} = 0, \quad \underline{sx3} + 1 \neq 0, \quad \underline{sx3} - 1 \neq 0]$$

only rotation allowed, $u1$ is zero function

Control Theory

$R = K\{u_1, \ldots, u_m\}$, $\quad U := \{u_1, \ldots, u_m\}$, $\quad S$ simple diff. system

Def.

$x \in U$ is *observable w.r.t.* $Y \subseteq U - \{x\}$

$$\iff \begin{cases} \exists\, p \in (E : q^\infty) - \{0\} \quad \text{s.t.} \\ p \in K\{Y\}[x] \qquad \text{(without derivatives of } x\text{)} \\ \text{initial of } p \notin (E : q^\infty), \quad \dfrac{\partial p}{\partial x} \notin (E : q^\infty) \end{cases}$$

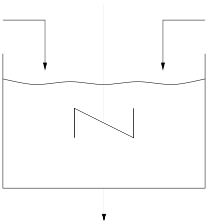
Def.

$Y \subseteq U$ is a *flat output*

$$\iff \begin{cases} (E : q^\infty) \cap K\{Y\} = \{0\} \\ \text{every } \quad x \in U - Y \quad \text{is observable w.r.t.} \quad Y \end{cases}$$

Example

Stirred tank:



$$\begin{cases} \dot{V}(t) = F_1(t) + F_2(t) - k\sqrt{V(t)} \\ \dot{\overline{c(t)\,V(t)}} = c_1\,F_1(t) + c_2\,F_2(t) - c(t)\,k\sqrt{V(t)} \end{cases}$$

H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, 1972.

Example

$R = \mathbb{Q}\{F_1, F_2, sV, c, c_1, c_2\}$, ranking $>$ s.t. $\{F_2, F_2\} \gg \{sV, c\} \gg \{c_1, c_2\}$

```
> with(DifferentialThomas):

> ivar := [t]:  dvar := [F1,F2,sV,c,c1,c2]:

> ComputeRanking(ivar, [[F1,F2],[sV,c],[c1,c2]]):

> L := [2*sV[t]*sV-F1-F2+k*sV,
c[t]*sV^2-c2*F2+c*k*sV-c1*F1+2*c*sV[t]*sV, c1[t], c2[t]]:

> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [2\, sV_1 sV_0 - F1_0 - F2_0 + k sV_0,$$
$$c_1 sV_0{}^2 - c2_0 F2_0 + c_0 k sV_0 - c1_0 F1_0 + 2\, c_0 sV_1 sV_0, \quad c1_1, \quad c2_1]$$

```
> TD := DifferentialThomasDecomposition(LL,
[sV[0],c1[0],c2[0]]);
```

$$TD := [DifferentialSystem, DifferentialSystem, DifferentialSystem]$$

Example

```
> Print(TD[1]);
```

$$[c2\,\underline{F1} - c1\,\underline{F1} + 2\,csV\,sV_t - 2\,c2\,sV\,sV_t + c_t sV^2 + cksV - c2\,ksV = 0,$$

$$c1\,\underline{F2} - c2\,\underline{F2} + 2\,csV\,sV_t - 2\,c1\,sV\,sV_t + c_t sV^2 + cksV - c1\,ksV = 0,$$

$$\underline{c1_t} = 0, \quad \underline{c2_t} = 0, \quad \underline{c2} \neq 0, \quad \underline{c1} \neq 0, \quad \underline{c1} - c2 \neq 0, \quad \underline{sV} \neq 0]$$

```
> collect(%[1], F1);
```

$$(c2 - c1)\,F1 + 2\,csV\,sV_t - 2\,c2\,sV\,sV_t + c_t sV^2 + cksV - c2\,ksV = 0$$

```
> collect(%%[2], F2);
```

$$(c1 - c2)\,F2 + 2\,csV\,sV_t - 2\,c1\,sV\,sV_t + c_t sV^2 + cksV - c1\,ksV = 0$$

$\Rightarrow$ F_1, F_2 observable with respect to $\{c, sV\}$

$(E : q^\infty) \cap \mathbb{Q}\{sV, c\} = \{0\}$ $\Rightarrow$ $\{c, sV\}$ is flat output

Example

```
> Print(TD[2]);
```

$$[c\underline{F1} - c2\,\underline{F1} + cF2 - c2\,F2 + c_t sV^2 = 0,$$
$$2\,c\underline{sV_t} - 2\,c2\,\underline{sV_t} + c_t sV + ck - c2\,k = 0, \quad \underline{c1} - c2 = 0, \quad \underline{c2_t} = 0,$$
$$\underline{c2} \neq 0, \quad \underline{c} - c2 \neq 0, \quad \underline{sV} \neq 0]$$

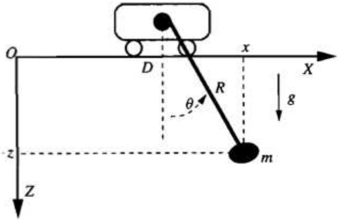
```
> Print(TD[3]);
```

$$[\underline{F1} + F2 - 2\,sV\,sV_t - k\,sV = 0, \quad \underline{c} - c2 = 0, \quad \underline{c1} - c2 = 0, \quad \underline{c2_t} = 0,$$
$$\underline{c2} \neq 0, \quad \underline{sV} \neq 0]$$

conditions $c_1 = c_2$ and $(c_1)_t = (c_2)_t = 0$ preclude control of the concentration in the tank

Example

2-D crane:



$$\begin{cases} m\,\ddot{x} & = & -T\,\sin\theta \\ m\,\ddot{z} & = & -T\,\cos\theta + m\,g \\ x & = & R\,\sin\theta + d \\ z & = & R\,\cos\theta \end{cases}$$

M. Fliess, J. Lévine, P. Martin, P. Rouchon, *Flatness and defect of non-linear systems: introductory theory and examples*, Internat. J. Control 61(6), 1327–1361, 1995.

Example

$\mathbb{Q}(m, g)\{T, s, c, d, R, x, z\}$

block ranking $>$ satisfying $\{T, s, c, d, R\} \gg \{x, z\}$

```
> with(DifferentialThomas):

> ivar := [t]:  dvar := [T,s,c,d,R,x,z]:

> ComputeRanking(ivar, [[T,s,c,d,R],[x,z]]):

> TD := DifferentialThomasDecomposition(
[m*x[2]+T[0]*s[0], m*z[2]+T[0]*c[0]-m*g,
x[0]-R[0]*s[0]-d[0], z[0]-R[0]*c[0], c[0]^2+s[0]^2-1],
[]);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

Example

```
> Print(TD[2]);
```

$$[z\underline{T} + m z_{t,t} R - m g R = 0, \quad z_{t,t} R \underline{s} - g R \underline{s} - z x_{t,t} = 0, \quad R \underline{c} - z = 0,$$
$$z_{t,t} \underline{d} - g \underline{d} + z x_{t,t} - x z_{t,t} + g x = 0,$$
$$z_{t,t}{}^2 \underline{R}^2 - 2\, g z_{t,t} \underline{R}^2 + g^2 \underline{R}^2 - z^2 x_{t,t}{}^2 - z^2 z_{t,t}{}^2 + 2\, g z^2 z_{t,t} - g^2 z^2 = 0,$$
$$\underline{z} \neq 0, \quad \underline{z_{t,t}} - g \neq 0, \quad \underline{x_{t,t}} \neq 0, \quad \underline{x_{t,t}}{}^2 + z_{t,t}{}^2 - 2\, g z_{t,t} + g^2 \neq 0]$$

```
> collect(%[5], R, factor);
```

$$(z_{t,t} - g)^2 R^2 - z^2 \left(x_{t,t}{}^2 + z_{t,t}{}^2 - 2\, g z_{t,t} + g^2\right) = 0$$

$\Rightarrow$ T, s, c, d, R observable with respect to $\{x, z\}$

$(E : q^\infty) \cap \mathbb{Q}\{x, z\} = \{0\}$ $\Rightarrow$ $\{x, z\}$ is flat output

Example

```
> Print(TD[1]);
```

$$[\underline{T} = 0, \quad R\underline{s} + d - x = 0, \quad R\underline{c} - z = 0, \quad \underline{d}^2 - 2\,x\underline{d} + x^2 - R^2 + z^2 = 0,$$
$$\underline{x_{t,t}} = 0, \quad \underline{z_{t,t}} - g = 0, \quad \underline{z} \neq 0, \quad \underline{R} \neq 0, \quad \underline{R} + z \neq 0, \quad \underline{R} - z \neq 0]$$

```
> Print(TD[3]);
```

$$[\underline{T} - mz_{t,t} + mg = 0, \quad \underline{s} = 0, \quad \underline{c} + 1 = 0, \quad \underline{d} - x = 0, \quad \underline{R} + z = 0,$$
$$\underline{x_{t,t}} = 0, \quad \underline{z} \neq 0]$$

```
> Print(TD[4]);
```

$$[\underline{T} + mz_{t,t} - mg = 0, \quad \underline{s} = 0, \quad \underline{c} - 1 = 0, \quad \underline{d} - x = 0, \quad \underline{R} - z = 0,$$
$$\underline{x_{t,t}} = 0, \quad \underline{z} \neq 0]$$

```
> Print(TD[5]);
```

$$[c\underline{T} - mg = 0, \quad g\underline{s} + x_{t,t}c = 0, \quad g^2\underline{c}^2 + {x_{t,t}}^2\underline{c}^2 - g^2 = 0, \quad \underline{d} - x = 0, \quad \underline{R} = 0,$$
$$\underline{z} = 0, \quad \underline{x_{t,t}} \neq 0, \quad \underline{x_{t,t}}^2 + g^2 \neq 0]$$

```
> Print(TD[6]);
```

$$[\underline{T} + mg = 0, \quad \underline{s} = 0, \quad \underline{c} + 1 = 0, \quad \underline{d} - x = 0, \quad \underline{R} = 0, \quad \underline{x_{t,t}} = 0, \quad \underline{z} = 0]$$

```
> Print(TD[7]);
```

$$[\underline{T} - mg = 0, \quad \underline{s} = 0, \quad \underline{c} - 1 = 0, \quad \underline{d} - x = 0, \quad \underline{R} = 0, \quad \underline{x_{t,t}} = 0, \quad \underline{z} = 0]$$

Example

$$
\left\{
\begin{array}{rcl}
-a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_1} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_1} - \left(\dfrac{\partial}{\partial x_2} a(x_2)\right) \xi_2(\mathbf{x}) + \dfrac{1}{2}\, a(x_2)\, (\nabla \cdot \xi(\mathbf{x})) & = & 0 \\
-a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_2} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_2} & = & 0 \\
-a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_3} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_3} - \dfrac{1}{2}\, (\nabla \cdot \xi(\mathbf{x})) & = & 0
\end{array}
\right.
$$

infinitesimal transformations of a Pfaffian system

$$
\frac{\partial}{\partial x_1} a(x_1, x_2, x_3) = 0, \qquad \frac{\partial}{\partial x_3} a(x_1, x_2, x_3) = 0.
$$

J.-F. Pommaret, A. Quadrat, *Formal obstructions to the controllability of partial differential control systems*, Proc. IMACS, Berlin, vol. 5, pp. 209–214, 1997.

Example

$R = \mathbb{Q}(x_1, x_2, x_3)\{\xi_1, \xi_2, \xi_3, a\}$ with diff. op. ∂_1, ∂_2, ∂_3 w.r.t. x_1, x_2, x_3

block ranking $>$ on R with blocks $\{\xi_1, \xi_2, \xi_3\}$, $\{a\}$

```
> with(DifferentialThomas):

> ivar := [x1,x2,x3]:  dvar := [xi1,xi2,xi3,a]:

> ComputeRanking(ivar, [[xi1,xi2,xi3],[a]]):
> L := [-a*xi1[x1]+xi3[x1]-a[x2]*xi2
+(1/2)*a*(xi1[x1]+xi2[x2]+xi3[x3]), -a*xi1[x2]+xi3[x2],
-a*xi1[x3]+xi3[x3] -(1/2)*(xi1[x1]+xi2[x2]+xi3[x3]),
a[x1], a[x3]]:

> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [-a_{0,0,0}\xi 1_{1,0,0} + \xi 3_{1,0,0} + 1/2\, a_{0,0,0}\left(\xi 1_{1,0,0} + \xi 2_{0,1,0} + \xi 3_{0,0,1}\right) - a_{0,1,0}\xi 2_{0,0,0}, \quad -a_{0,0,0}\xi 1_{0,1,0} + \xi 3_{0,1,0}, \quad -a_{0,0,0}\xi 1_{0,0,1} + 1/2\,\xi 3_{0,0,1} - 1/2\,\xi 1_{1,0,0} - 1/2\,\xi 2_{0,1,0}, \quad a_{1,0,0}, \quad a_{0,0,1}]$$

```
> TD := DifferentialThomasDecomposition(LL, []);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

Example

```
> Print(TD[1]);
```

$$[a\underline{\xi 1_{x2}} - \xi 3_{x2} = 0, \quad a^2\underline{\xi 1_{x3}} + \xi 3_{x1} = 0, \quad \underline{\xi 2} = 0,$$
$$a\underline{\xi 1_{x1}} - 2\,\xi 3_{x1} - a\xi 3_{x3} = 0, \quad \underline{a_{x1}} = 0, \quad \underline{a_{x3}} = 0, \quad \underline{a} \neq 0]$$

parameter $a = a(x_2)$ non-zero, but otherwise arbitrary

```
> Print(TD[2]);
```

$$[a\underline{\xi 1_{x2}} - \xi 3_{x2} = 0, \quad a^2\underline{\xi 1_{x3}} + a\xi 2_{x2} - a_{x2}\xi 2 + \xi 3_{x1} = 0,$$
$$a\underline{\xi 1_{x1}} - a\xi 2_{x2} + 2\,a_{x2}\xi 2 - 2\,\xi 3_{x1} - a\xi 3_{x3} = 0, \quad \underline{a_{x1}} = 0,$$
$$\underline{a_{x2,x2}} = 0, \quad \underline{a_{x2,x3}} = 0, \quad \underline{a_{x3}} = 0, \quad \underline{a} \neq 0, \quad \underline{\xi 2} \neq 0]$$

parameter subject to $a_{x_2,x_2} = 0$

```
> Print(TD[3]);
```

$$[\underline{\xi 1_{x1,x1}} + \xi 2_{x1,x2} = 0, \quad \underline{\xi 1_{x1,x2}} + \xi 2_{x2,x2} = 0, \quad \underline{\xi 3_{x1}} = 0, \quad \underline{\xi 3_{x2}} = 0,$$
$$\underline{\xi 1_{x1}} + \xi 2_{x2} - \xi 3_{x3} = 0, \quad \underline{a} = 0]$$

parameter $a = 0$

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