A propos d'un groupe d'associateurs

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Outline

INTRODUCTION

Zeta functions with several complex indices

$$\mathcal{H}_r := \{(s_1,\ldots,s_r) \in \mathbb{C}^r | \forall m = 1,\ldots,r, \Re(s_1)+\ldots+\Re(s_m) > m\}, r \in \mathbb{N}_+.$$
$$\zeta(s_1,\ldots,s_r) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r} \quad \text{converges for} \quad (s_1,\ldots,s_r) \in \mathcal{H}_r.$$

For $n \in \mathbb{N}, z \in \mathbb{C}, |z| < 1, (s_1,\ldots,s_r) \in \mathbb{C}^r$, let us define the following functions
$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \quad \text{and} \quad \frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{n\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(n) z^n.$$
Hence, from a theorem by Abel, one has
$$\forall (s_1,\ldots,s_r) \in \mathcal{H}_r, \quad \zeta(s_1,\ldots,s_r) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z).$$
$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathcal{H}_r\cap\mathbb{N}^r, r\in\mathbb{N}}.$$
These values do appear in the *regularization* of solutions of the following differential equation with noncommutative indeterminates in $X = \{x_0, x_1\}$
$$(DE) \quad dG = MG, \text{ with } M = \omega_0 x_0 + \omega_1 x_1, \omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1-z}.$$

Drinfel'd stated that (DE) has a unique solution G_0 (resp. G_1), being group-like series, s.t. $G_0(z) \sim_0 e^{x_0 \log(z)}$ (resp. $G_1(z) \sim_1 e^{-x_1 \log(1-z)}$). There is then a unique series $\Phi_{KZ} \in \mathbb{R}\langle\langle X\rangle\rangle$, $\Delta_{ш}(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ}$, such that $G_0 = G_1 \Phi_{KZ}$. This series is called Drinfel'd associator.

Indexing by words

Introducing $Y = \{y_k\}_{k\geq 1}$, $Y_0 = Y \cup \{y_0\}$ and using the correspondences

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r \quad\leftrightarrow\quad y_{s_1}\ldots y_{s_r}\in Y^* \quad \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} \quad x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1,$$

$$(s_1,\ldots,s_r)\in\mathbb{N}^r \quad\leftrightarrow\quad y_{s_1}\ldots y_{s_r}\in Y_0^*,$$

we denote $\mathrm{H}_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{s_1,\ldots,s_r}$ and $\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} := \mathrm{Li}_{s_1,\ldots,s_r}$,

$\mathrm{H}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{-s_1,\ldots,-s_r}$ and $\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{Li}_{-s_1,\ldots,-s_r}$,

and also $\zeta(y_{s_1}\ldots y_{s_r}) := \zeta(s_1,\ldots,s_r) =: \zeta(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1)$,

$\zeta^-(y_{s_1}\ldots y_{s_r}) := \zeta(-s_1,\ldots,-s_r)$.

The polylogarithms can be viewed as iterated integrals, w.r.t. ω_0,ω_1 and associated to words in X^* : $\mathrm{Li}_{s_1,\ldots,s_r}(z) = \alpha_0^z(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1)$, where

$$\alpha_{z_0}^z(1_{X^*}) = 1_\Omega \quad\text{and}\quad \alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k),$$

where $(z_0, z_1\ldots,z_k,z)$ is a subdivision of the path $z_0 \rightsquigarrow z$ in the simply connected domain $\Omega := \widetilde{\mathbb{C}-\{0,1\}}$ and $1_\Omega:\Omega\to\mathbb{C}$, mapping z to 1.

$\theta_0 := z\partial_z, \theta_1 := (1-z)\partial_z$ and ι_0,ι_1 such that $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$ (*i.e.* the sections of them, taking primitives for the corresponding differential operators). Then

$\mathrm{Li}_{-s_1,\ldots,-s_r} = (\theta_0^{t_1+1}\iota_1\ldots\theta_0^{t_r+1}\iota_1)1_\Omega$ and $\mathrm{Li}_{s_1,\ldots,s_r} = (\iota_0^{s_1-1}\iota_1\ldots\iota_0^{s_r-1}\iota_1)1_\Omega$.

Noncommutative, co-commutative bialgebras

X and Y are ordered, respectively, by $x_1 > x_0$ and $y_1 > y_2 > \ldots$.

$\mathcal{L}ynX, \{S_l\}_{l\in\mathcal{L}ynX}$: pure transcendence bases of[1] $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})$,

$\mathcal{L}ynY, \{\Sigma_l\}_{l\in\mathcal{L}ynY}$: pure transcendence bases of[2] $(\mathbb{C}\langle Y\rangle, ⧺, 1_{Y^*})$,

$\{P_l\}_{l\in\mathcal{L}ynX}, \{\Pi_l\}_{l\in\mathcal{L}ynY}$: homogeneous (graded) bases of Lie algebras of primitive elements, for respectively $\Delta_{⧢}, \Delta_{⧺}$,

- in the concatenation-shuffle bialgebra $(\mathbb{C}\langle X\rangle, \mathtt{conc}, \Delta_{⧢}, 1_{X^*}, \mathtt{e})$,

$$\mathcal{D}_X := \sum_{w\in X^*} w\otimes w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{S_l\otimes P_l} \quad \text{(MRS-factorization)}.$$

- in the concatenation-stuffle bialgebra $(\mathbb{C}\langle Y\rangle, \mathtt{conc}, \Delta_{⧺}, 1_{Y^*}, \mathtt{e})$,

$$\mathcal{D}_Y := \sum_{w\in Y^*} w\otimes w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes \Pi_l} \quad (⧺ - \text{extended MRS-factorization}).$$

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[1] For $x, y \in X, u, v \in X^*$, $u ⧢ 1_{X^*} = 1_{X^*} ⧢ u = u$ and $xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v)$, or equivalently $\Delta_{⧢}(x) = x\otimes 1_{X^*} + 1_{X^*}\otimes x$ (*i.e.* letters are primitive, for $\Delta_{⧢}$).

[2] For $y_i, y_j \in Y, u, v \in Y^*$, $u ⧺ 1_{Y^*} = 1_{Y^*} ⧺ u = u$ and $y_i u ⧺ y_j v = y_i(u ⧺ y_j v) + y_j(y_i u ⧺ v) + y_{i+j}(u ⧺ v)$, or equivalently $\Delta_{⧺}(y_i) = y_i\otimes 1_{Y^*} + 1_{Y^*}\otimes y_i + \sum_{k+l=i} y_k\otimes y_k$ (*i.e.* y_1 is primitive, for $\Delta_{⧺}$).

First structures of polylogarithms and harmonic sums

1. Completed with $\mathrm{Li}_{x_0^k}(z) := \log^k(z)/k!$, $\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathbb{C}$-linearly independent. Hence, the following morphism of algebras is injective
$$\mathrm{Li}_\bullet : (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \to (\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \quad u \mapsto \mathrm{Li}_u .$$
Thus, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_{S_l}\}_{l\in\mathcal{L}ynX}$) are algebraically independent.
2. The following morphism of algebras is injective
$$\mathrm{H}_\bullet : (\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*}) \to (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \quad u \mapsto \mathrm{H}_u.$$
Hence, $\{\mathrm{H}_w\}_{w\in Y^*}$ is $\mathbb{C}$-linearly independent. It follows that, $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{\mathrm{H}_{\Sigma_l}\}_{l\in\mathcal{L}ynY}$) are algebraically independent.
3. $\zeta : \dfrac{(\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ш, 1_{X^*})}{(\mathbb{Q}1_{Y^*} \oplus (Y - \{y_1\})\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*})} \twoheadrightarrow (\mathcal{Z}, ., 1)$ such that, for any $l_1, l_2 \in \mathcal{L}ynX - X$, $\zeta(l_1 ш l_2) = \zeta((\pi_Y l_1) ⧢ (\pi_Y l_2)) = \zeta(l_1)\zeta(l_2)$.
4. There exists, at least, an associative law of algebra $\top$, in $\mathbb{Q}\langle Y_0\rangle$, (not dualizable) such that the following morphism is onto
$$\mathrm{Li}^-_\bullet \quad : \quad (\mathbb{Q}\langle Y_0\rangle, \top) \quad \to \quad (\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*}, .), \quad w \mapsto \mathrm{Li}^-_w,$$
and $\ker \mathrm{Li}^-_\bullet = \mathbb{Q}\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}$.

   Moreover, if $\top' : \mathbb{Q}\langle Y_0\rangle \times \mathbb{Q}\langle Y_0\rangle \to \mathbb{Q}\langle Y_0\rangle$ is a law such that $\mathrm{Li}^-_\bullet$ is a morphism for $\top'$ and $(1_{Y_0^*}\top'\mathbb{Q}\langle Y_0\rangle) \cap \ker(\mathrm{Li}^-_\bullet) = \{0\}$ then $\top' = g \circ \top$, where $g \in GL(\mathbb{Q}\langle Y_0\rangle)$ such that $\mathrm{Li}^-_\bullet \circ g = \mathrm{Li}^-_\bullet$.

SINGULAR AND ASYMPTOTIC EXPANSIONS

Noncommutative series and first Abel like theorem

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w\, w = (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l}, \quad Z_{\sqcup\!\sqcup} := \prod_{l\in\mathcal{L}ynX - X}^{\searrow} e^{\zeta(S_l)P_l},$$

$$\mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = (\mathrm{H}_\bullet \otimes \mathrm{Id})\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\Sigma_l}\Pi_l}, \quad Z_{\uplus} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l}.$$

L is solution of (DE) satisfying $\mathrm{L}(z) \sim_0 e^{x_0 \log(z)}$. One has $\mathrm{L}(z) \sim_1 e^{-x_1 \log(1-z)} Z_{\sqcup\!\sqcup}$.

Theorem (HNM, 2005)

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y \mathrm{L}(z) = \lim_{n\to\infty} e^{\sum_{k\geq 1} \mathrm{H}_{y_k}(n)(-y_1)^k/k}\mathrm{H}(n) = \pi_Y Z_{\sqcup\!\sqcup} .$$

For $w \in X^* x_1$, there exists $a_i, b_{i,j} \in \mathcal{Z}$ and $\alpha_i, \beta_{i,j}, \gamma_{\pi_Y w} \in \mathcal{Z}[\gamma]$ such that

$$\mathrm{Li}_w(z) \underset{z\to 1}{\asymp} \sum_{i=1}^{|w|} a_i \log^i(1-z) + \langle Z_{\sqcup\!\sqcup} | w\rangle + \sum_{i,j\in\mathbb{N}_+} b_{i,j}(1-z)^j \log^i(1-z),$$

$$\mathrm{H}_{\pi_Y w}(n) \underset{n\to+\infty}{\asymp} \sum_{i=1}^{(w)} \alpha_i \log^i(n) + \gamma_{\pi_Y w} + \sum_{i,j\in\mathbb{N}_+} \beta_{i,j} \frac{\log^i(n)}{n^j}.$$

Let $Z_\gamma := \sum_{w\in Y^*} \gamma_w w$. Then Z_γ is group-like, for $\Delta_{\uplus}$. By the extended MRS-factorization, one has $Z_\gamma = e^{\gamma y_1} Z_{\uplus}$ and then, by the Abel like theorem, one deduces $(Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup} \Leftrightarrow Z_{\uplus} = B'(y_1)\pi_Y Z_{\sqcup\!\sqcup})$, where $B(y_1) = e^{\gamma y_1 - \sum_{k\geq 2}\zeta(k)(-y_1)^k/k}$ and $B'(y_1) = e^{-\sum_{k\geq 2}\zeta(k)(-y_1)^k/k}$.

Actions of the Galois differential group

$$\overline{\mathrm{L}} := \mathrm{L}e^{C},\quad \overline{Z}_{\sqcup\!\sqcup} := Z_{\sqcup\!\sqcup}\, e^{C},\quad \overline{\mathrm{H}} = \sum_{w\in Y^*} \overline{\mathrm{H}}_w w,\quad \overline{Z}_{\gamma} := \sum_{w\in Y^*} \overline{\gamma}_w w,$$

where $e^{C} \in \{e^{C}\}_{C\in\mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle} = \mathrm{Gal}_{\mathbb{C}}(DE)$ and, for any $w \in Y^*$, letting $v = \pi_X w \in X^* x_1$, one has

$$\sum_{n\geq 0} \overline{\mathrm{H}}_w(n) z^n = \frac{\langle \overline{\mathrm{L}}(z) | v\rangle}{1-z} \quad \text{and} \quad \overline{\gamma}_w := \mathrm{f.p.}\overline{\mathrm{H}}_w(n), \text{ for } \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.$$

$$\overline{\mathrm{L}}(z) \sim_1 e^{-x_1\log(1-z)}\overline{Z}_{\sqcup\!\sqcup} \quad \text{and} \quad \overline{\mathrm{H}}(n) \sim_{+\infty} e^{-\sum_{k\geq 1}\mathrm{H}_{y_k}(n)(-y_1)^k/k}\pi_Y \overline{Z}_{\sqcup\!\sqcup}.$$

It follows then an extended Abel like theorem :

$$\lim_{z\to 1} e^{y_1\log(1-z)}\pi_Y\overline{\mathrm{L}}(z) = \lim_{n\to+\infty} e^{\sum_{k\geq 1}\mathrm{H}_{y_k}(n)(-y_1)^k/k}\overline{\mathrm{H}}(n) = \pi_Y \overline{Z}_{\sqcup\!\sqcup}.$$

Therefore, one has a bridge equation $\overline{Z}_{\gamma} = B(y_1)\pi_Y \overline{Z}_{\sqcup\!\sqcup}$.

$\overline{\mathrm{L}}$ is solution of (DE) satisfying $\overline{\mathrm{L}}(z) \sim_0 e^{x_0\log(z)}e^{C}$.

Thus, L is unique, satisfying[3] $\mathrm{L}(z) \sim_0 e^{x_0\log(z)}$, and $\Phi_{KZ} = Z_{\sqcup\!\sqcup}$ is also unique.

Theorem (HNM, 2009)

For $\mathbb{Q} \subset A \subset \mathbb{C}$, let[4] $dm(A) := \{Z_{\sqcup\!\sqcup}\, e^{C}\}_{C\in\mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C|x_0\rangle = \langle e^C|x_1\rangle = 0}$.
If $\overline{Z}_{\sqcup\!\sqcup} \in dm(A)$ *then* $(\overline{Z}_{\gamma} = B(y_1)\pi_Y\overline{Z}_{\sqcup\!\sqcup} \Leftrightarrow \overline{Z}_{\pm\!\!\sqcup} = B'(y_1)\pi_Y\overline{Z}_{\sqcup\!\sqcup})$.
Hence, if $\gamma \notin A$ then γ is transcendent over A.

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[3] See also Duchamp's talk.

[4] $dm(A) = \mathrm{Gal}_A^{\geq 2}(DE)$ is a strict normal sub-group of $\mathrm{Gal}_A(DE)$.

Homogenous polynomials relations among local coordinates

$$Z_\gamma = B(y_1)\pi_Y Z_{⧢}$$

	Polynomial relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Polynomial relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

(Bùi's, Duchamp, HNM, 2015)

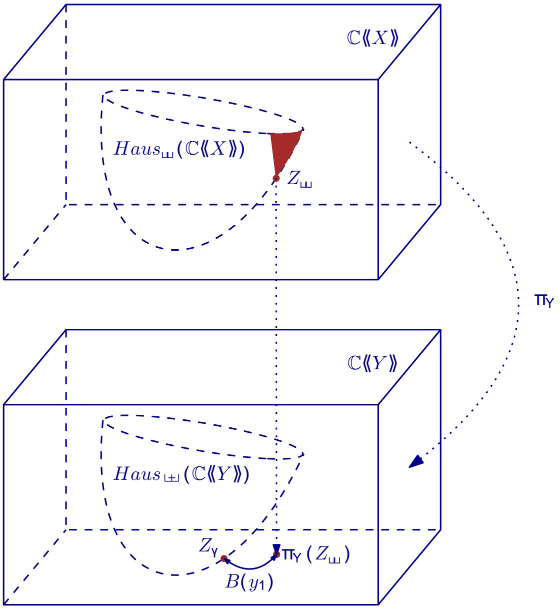
Noetherian rewriting system & irreducible coordinates[5]

	Rewriting system on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting system on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) \to \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) \to \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) \to \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) \to \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) \to \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) \to 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) \to -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) \to -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) \to \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) \to -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) \to \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) \to \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) \to \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) \to \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) \to \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) \to \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) \to \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) \to \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) \to -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) \to \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) \to 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) \to \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) \to \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) \to -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) \to \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) \to \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) \to \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) \to \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) \to \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) \to \frac{8}{35}\zeta(S_{x_0x_1})^3$

(Bùi's, Duchamp, HNM, 2015)

[5] The set of irreducible coordinates forms algebraic generator system for $\mathcal{Z}$.

Illustration of $Z_\gamma = B(y_1)\pi_Y Z_{⧢}$



Integro differential algebra and second Abel like theorem

1. For any $w \in Y_0^*$, Li_w^- (resp. H_w^-) is a polynomial in $\mathbb{Z}[(1-z)^{-1}]$ (resp. $\mathbb{Q}[n]$), of valuation 1 and of degree $d := |w| + (w)$. Hence, $\mathrm{Li}_w^-(z) \sim_1 B_w^-(1-z)^{-d}$ and $\mathrm{H}_w^-(n) \sim_\infty C_w^- n^d$, where
$$B_w^- = d! C_w^- \in \mathbb{Z} \quad \text{and} \quad C_w^- = \prod_{w=uv, v \neq 1_{Y_0^*}} ((v) + |v|)^{-1} \in \mathbb{Q}.$$
2. The families $\{\mathrm{Li}_{y_k}^-\}_{k \geq 0}$ and $\{\mathrm{H}_{y_k}^-\}_{k \geq 0}$ are $\mathbb{Q}$-linearly independent.
3. Let $\mathcal{C} := (\mathbb{C}[z, z^{-1}, (1-z)^{-1}], \partial_z)$. Then the algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}$ ($\cong \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}$) is stable under the operators[6] $\{\theta_0, \theta_1, \iota_0, \iota_1\}$.
4. The bi-integro differential algebra $(\mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, \theta_0, \theta_1, \iota_0, \iota_1)$ is closed under the action of the group of transformations, $\mathcal{G}$, generated by $\{z \mapsto 1-z, z \mapsto z^{-1}\}$, permuting singularities in $\{0, 1, +\infty\}$:
$$\forall h \in \mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g) \in \mathcal{C}\{\mathrm{Li}_w\}_{w \in X^*}.$$

Theorem (Duchamp, HNM, Ngô, 2015)

$$\mathrm{L}^- := \sum_{w \in Y_0^*} \mathrm{Li}_w^- \, w, \quad \mathrm{H}^- := \sum_{w \in Y_0^*} \mathrm{H}_w^- w, \quad C^- := \sum_{w \in Y_0^*} C_w^- w.$$
$$\lim_{z \to 1} h^{\odot -1}((1-z)^{-1}) \odot \mathrm{Li}^-(z) = \lim_{n \to +\infty} g^{\odot -1}(n) \odot \mathrm{H}^-(n) = C^-,$$
where $h(t) = \sum_{w \in Y_0^*} ((w) + |w|)! t^{(w)+|w|} w$ *and* $g(t) = \sum_{w \in Y_0^*} t^{(w)+|w|} w$.

Moreover, H^- *and* C^- *are group-like, respectively, for* $\Delta_{\boxplus}$ *and* $\Delta_{\text{ш}}$.

POLYLOGARITHMS AND HARMONIC SUMS INDEXED BY NONCOMMUTATIVE RATIONAL SERIES

Rational series ($\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$)–Exchangeable series ($\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$)

Theorem (Schützenberger, 1961)

$R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ iff there is a linear representation, (ν,μ,η) of dimension $n>0$, i.e. $\nu \in M_{1,n}(\mathbb{C})$, $\eta \in M_{n,1}(\mathbb{C})$ and $\mu : X^* \to M_{n,n}(\mathbb{C})$ such that

$$R = \nu\left(\sum_{w\in X^*} \mu(w)\; w\right)\eta = \nu((\mathrm{Id}\otimes\mu)\mathcal{D}_X)\eta.$$

Theorem (HNM, 1995)

For any $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, the series $\sum_{w\in X^*}\langle R|w\rangle\alpha_{z_0}^{z}(w) =: \langle R\|S_{z_0\rightsquigarrow z}\rangle$ is convergent, where $\sum_{w\in X^*}\alpha_{z_0}^{z}(w)w$ denotes the Chen series $S_{z_0\rightsquigarrow z}$, and

$$\forall U,V \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle,\quad \langle U ⧢ V\|S_{z_0\rightsquigarrow z}\rangle = \langle U\|S_{z_0\rightsquigarrow z}\rangle\langle V\|S_{z_0\rightsquigarrow z}\rangle.$$

Moreover, letting (ν,μ,η) be a linear representation of R, one has

$$\langle R\|S_{z_0\rightsquigarrow z}\rangle = \nu((\alpha_{z_0}^{z}\otimes\mu)\mathcal{D}_X)\eta = \nu\left(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^{z}(S_l)\mu(P_l)}\right)\eta.$$

The power series S belongs to $\mathbb{C}_{\mathrm{exc}}\langle\langle X\rangle\rangle$, iff

$$(\forall u,v \in X^*)((\forall x\in X)(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle.$$

If $S = \sum_{i_0,i_1\geq 0} s_{i_0,i_1} x_0^{i_0} ⧢ x_1^{i_1}$ then $\langle S\|S_{z_0\rightsquigarrow z}\rangle = \sum_{i_0,i_1\geq 0} s_{i_0,i_1}\dfrac{(\alpha_{z_0}^{z}(x_0))^{i_0}}{i_0!}\dfrac{(\alpha_{z_0}^{z}(x_1))^{i_1}}{i_1!}.$

Polylogarithms, harmonic sums and rational series

Lemma (Duchamp, HNM, Ngô, 2016)

1. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle := \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \bigcap \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle = \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$.
2. *For any* $x \in X$, *one has* $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* \sqcup\!\sqcup\, \mathbb{C}\langle x\rangle | a \in \mathbb{C}\}$.
3. *The family* $\{x_0^*, x_1^*\}$ *is algebraically independent over* $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ *within* $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*})$.
4. *The module* $(\mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ *is* $\mathbb{C}\langle X\rangle$*-free and* $\{(x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ *forms a* $\mathbb{C}\langle X\rangle$*-basis of it.*

   Hence, $\{w \sqcup\!\sqcup\, (x_0^*)^{\sqcup\!\sqcup\, k} \sqcup\!\sqcup\, (x_1^*)^{\sqcup\!\sqcup\, l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}_{w\in X^*}$ *is a* $\mathbb{C}$*-basis of it.*

Theorem (extension of $\mathrm{Li}_\bullet$, Duchamp, HNM, Ngô, 2016)

$\mathrm{Li}_\bullet : (\mathbb{C}[x_0^*, x_1^*, (-x_0)^*] \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \twoheadrightarrow (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1_\Omega), R \mapsto \mathrm{Li}_R$.
$\mathrm{Li}_\bullet$ *is surjective and* $\ker \mathrm{Li}_\bullet$ *is the shuffle ideal generated by* $x_0^* \sqcup\!\sqcup\, x_1^* - x_1^* + 1$.

Example (of polylogarithms indexed by rational series)

Since, for any $n \in \mathbb{N}, a, b \in \mathbb{C}$, one has

$$\langle (bx_1)^* \| S_{0\rightsquigarrow z}\rangle = (1-z)^{-b} \quad \text{and} \quad \langle (ax_0)^* \| S_{1\rightsquigarrow z}\rangle = z^a$$

then $\mathrm{Li}_{x_0^*}(z) = z$, $\mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}$, $\mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}$.

Indexing polylogarithms by rational series (1/2)

$$\mathrm{Li}_{-s_1,\ldots,-s_r} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\ldots\sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots \binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r}(\theta_0^{k_1}\mathrm{Li}_0)\ldots(\theta_0^{k_r}\mathrm{Li}_0),$$

$$\theta_0^{k_i}(\mathrm{Li}_0(z)) = \frac{1}{1-z}\sum_{j=1}^{k_i} S_2(k_i,j)j!(\mathrm{Li}_0(z))^j, \quad \text{for } k_i>0,$$

where $\mathrm{Li}_0(z)=z(1-z)^{-1}$, $S_2(k_i,j)$ are the Stirling numbers of second kind.

Lemma (Encoding polylogarithms by rational series)

$\mathrm{Li}_{-s_1,\ldots,-s_r}=\mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}$, *where* $R_{y_{s_1}\ldots y_{s_r}}\in(\mathbb{Z}[x_1^*],\,ш\,,1_{X^*})$ *given by*

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{\substack{k_1=0,\ldots,s_1;k_2=0,\ldots,s_1+s_2-k_1;\ldots;\\k_r=0,\ldots,(s_1+\ldots+s_r)-(k_1+\ldots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\ldots\binom{\sum_{i=1}^r s_i-\sum_{i=1}^{r-1}k_i}{k_r}\rho_{k_1}\,ш\,\ldots\,ш\,\rho_{k_r},$$

$$\rho_{k_i} = \begin{cases} x_1^*-1_{X^*}, & \textit{if}\quad k_i=0,\\ x_1^*\,ш\,\sum_{j=1}^{k_i}S_2(k_i,j)j!(x_1^*-1_{X^*})^{ш\, j}, & \textit{if}\quad k_i>0.\end{cases}$$

By linearity, $R_\bullet$ is extended over $\mathbb{Z}\langle Y_0\rangle$.

Indexing polylogarithms by rational series (2/2)

Theorem (restriction of $\mathrm{Li}_\bullet$)

The restriction $\mathrm{Li}_\bullet : (\mathbb{Z}[x_1^*], ш, 1_{X^*}) \to (\mathbb{Z}[(1-z)^{-1}], ., 1_\Omega)$ *is bijective and the family* $\{\mathrm{Li}_{R_{y_k}}\}_{k\geq 0}$ *is a* $\mathbb{Z}$*-basis of the image. Hence,* $\forall k \geq 0$, $\exists! R_{y_k} \in \mathbb{Z}[x_1^*]$ *s.t.* $\mathrm{Li}_{R_{y_k}} = \mathrm{Li}_{-k}$. *Moreover,* $R_{y_0} = x_1^* - 1_{X^*}$ *and*

$$\forall k \geq 1, \quad R_{y_k} = x_1^* ш \left(\sum_{i=0}^{k} i! S_2(k,i)(x_1^* - 1)^{ш\, i}\right),$$

$$((x_1)^*)^{ш\, k} = 1_{X^*} + R_{y_0} + \sum_{j=2}^{k} \frac{S_1(k,j)}{(k-1)!} R_{y_{j+1}},$$

where $S_1(k,i)$ *and* $S_2(k,j)$ *are Stirling numbers of first and second kind.*

Corollary

The morphism $R_\bullet : (\mathbb{Z}\langle Y\rangle, ⧢_{\!\!-}, 1_{Y_0^*}) \to (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ *is bijective. Hence, for any* $l \in \mathcal{L}ynY$, *there exists a unique polynomial* $p \in \mathbb{Z}[t]$ *of degree* $(l) + |l|$ *and of valuation* 1 *such that*

$$\begin{aligned} R_l &= \check{p}(x_1^*) &\in& (\mathbb{Z}[x_1^*], ш, 1_{X^*}),\\ \mathrm{Li}_{R_l}(z) &= p(e^{-\log(1-z)}) &\in& (\mathbb{Z}[e^{-\log(1-z)}], ., 1),\\ \mathrm{H}_{\pi_Y R_l}(n) &= \tilde{p}((n)_\bullet) &\in& (\mathbb{Q}[(n)_\bullet], ., 1), \end{aligned}$$

where $(n)_\bullet : \mathbb{N} \to \mathbb{N}, i \mapsto n(n-1)\ldots(n-i+1)$, $\tilde{p}$ *is the exponential transformed of* p *and* p *is obtained as the exponential transformed of* $\check{p}$.

Constants $\{\gamma_{-s_1,\ldots,-s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}^r, r\in\mathbb{N}}$

Theorem (extended double regularization)

$$\zeta((tx_1)^*) = \langle Z_{ш} \| (tx_1)^* \rangle = 1,$$

$$\gamma_{\pi_Y(tx_1)^*} = \langle Z_\gamma \| (ty_1)^* \rangle = \exp\Big(\gamma t - \sum_{n\geq 2} \zeta(n) \frac{(-t)^n}{n}\Big) = \frac{1}{\Gamma(1+t)}.$$

Corollary

For any $l \in \mathcal{L}ynY$, there exists a unique polynomial $p \in \mathbb{Z}[t]$ of degree $(l) + |l|$ and of valuation 1 such that $R_l = \check{p}(x_1^*) \in (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ and

$$\zeta(R_l) = p(1) \in \mathbb{Z} \quad \text{and} \quad \gamma_{\pi_Y R_l} = \tilde{p}(1) \in \mathbb{Q},$$

where $\tilde{p}$ is the exponential transformed of p and p is obtained as the exponential transformed of $\check{p}$.

Example

$$\mathrm{Li}_{-1,-1} = -\mathrm{Li}_{x_1^*} + 5\,\mathrm{Li}_{(2x_1)^*} - 7\,\mathrm{Li}_{(3x_1)^*} + 3\,\mathrm{Li}_{(4x_1)^*},$$
$$\mathrm{Li}_{-2,-1} = \mathrm{Li}_{x_1^*} - 11\,\mathrm{Li}_{(2x_1)^*} + 31\,\mathrm{Li}_{(3x_1)^*} - 33\,\mathrm{Li}_{(4x_1)^*} + 12\,\mathrm{Li}_{(5x_1)^*},$$
$$\mathrm{Li}_{-1,-2} = \mathrm{Li}_{x_1^*} - 9\,\mathrm{Li}_{(2x_1)^*} + 23\,\mathrm{Li}_{(3x_1)^*} - 23\,\mathrm{Li}_{(4x_1)^*} + 8\,\mathrm{Li}_{(5x_1)^*},$$
$$\mathrm{H}_{-1,-1} = -\mathrm{H}_{\pi_Y(x_1^*)} + 5\mathrm{H}_{\pi_Y((2x_1)^*)} - 7\mathrm{H}_{\pi_Y((3x_1)^*)} + 3\mathrm{H}_{\pi_Y((4x_1)^*)},$$
$$\mathrm{H}_{-2,-1} = \mathrm{H}_{\pi_Y(x_1^*)} - 11\mathrm{H}_{\pi_Y((2x_1)^*)} + 31\mathrm{H}_{\pi_Y((3x_1)^*)} - 33\mathrm{H}_{\pi_Y((4x_1)^*)} + 12\mathrm{H}_{\pi_Y((5x_1)^*)},$$
$$\mathrm{H}_{-1,-2} = \mathrm{H}_{\pi_Y(x_1^*)} - 9\mathrm{H}_{\pi_Y((2x_1)^*)} + 23\mathrm{H}_{\pi_Y((3x_1)^*)} - 23\mathrm{H}_{\pi_Y((4x_1)^*)} + 8\mathrm{H}_{\pi_Y((5x_1)^*)}.$$

Therefore, $\zeta(-1,-1) = 0$, $\zeta(-2,-1) = -1$, $\zeta(-1,-2) = 0$, and

$$\gamma_{-1,-1} = -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = 11/24,$$
$$\gamma_{-2,-1} = \Gamma^{-1}(2) - 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -73/120,$$
$$\gamma_{-1,-2} = \Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -67/120.$$

Candidates for associators with rational coefficients

$$\Upsilon := ((\mathrm{H}_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y \quad \text{and} \quad \Lambda := ((\mathrm{Li}_\bullet \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X,$$
$$Z_\gamma^- := ((\gamma_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y \quad \text{and} \quad Z_{\shuffle}^- := ((\zeta \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X,$$

where the morphism of algebras $\hat{\pi}_Y$ is defined, over an algebraic basis, by $\hat{\pi}_Y(x_0) = x_0$ (such that $\mathrm{Li}_{R_{\hat{\pi}_Y x_0}}(z) = \log(z)$ and then $\zeta(R_{\hat{\pi}_Y x_0}) = 0$) and, for any $l \in \mathcal{L}ynX - \{x_0\}, \hat{\pi}_Y S_l = \pi_Y S_l$.

Hence, $Z_\gamma^- \in \mathbb{Q}\langle\langle Y\rangle\rangle$ and $Z_{\shuffle}^- \in \mathbb{Z}\langle\langle X\rangle\rangle$. In particular,

$$\langle Z_\gamma^- | y_1\rangle = -1/2 \text{ and } \langle Z_{\shuffle}^- | x_1\rangle = \langle Z_{\shuffle}^- | x_0\rangle = 0.$$

Theorem (candidates for associators with rational coefficients)

$$\Delta_{\uplus}(\Upsilon) = \Upsilon \otimes \Upsilon \quad \textit{and} \quad \Delta_{\shuffle}(\Lambda) = \Lambda \otimes \Lambda,$$
$$\Delta_{\uplus}(Z_\gamma^-) = Z_\gamma^- \otimes Z_\gamma^- \quad \textit{and} \quad \Delta_{\shuffle}(Z_{\shuffle}^-) = Z_{\shuffle}^- \otimes Z_{\shuffle}^-,$$

and all constant terms are 1*. It follows then*

$$\Upsilon = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad \Lambda = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\hat{\pi}_Y S_l}} P_l} \sim_0 e^{x_0 \log(z)},$$
$$Z_\gamma^- = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y R_{\Sigma_l}} \Pi_l} \quad \textit{and} \quad Z_{\shuffle}^- = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\zeta_{\shuffle}(R_{\hat{\pi}_Y S_l}) P_l}.$$

Moreover, $\Lambda \in (\mathrm{span}_{\mathbb{C}}\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle}, \theta_0, \iota_0, \theta_1, \iota_1)\langle\langle X\rangle\rangle$ and, for any $g \in \mathcal{G}$, there exists a letter substitution, σ_g, and a Lie series, $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$, such that $\Lambda(g) = \sigma_g(\Lambda)e^C$.

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THANK YOU FOR YOUR ATTENTION