Decomposition of Low Rank Multi-Symmetric Tensor

HARMOUCH Jouhayna

Lebanese University, Lebanon
UCA, INRIA, France

February 1, 2018

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The Analysis of contents of Web Pages Model

- * Collection: n documents
- * What is the topic of a document?
- * Each document is represented by a vector $c \in \mathbb{R}^n$ where each component is the occurence of a word drawn from a vocabulary which contains n words. This is a sparse vector.

* We compute the 2-order tensors associated to all c vectors $M_{2c} = \frac{l}{2}(c \otimes c - diag(c))$, and the mean M_2 of all of them.

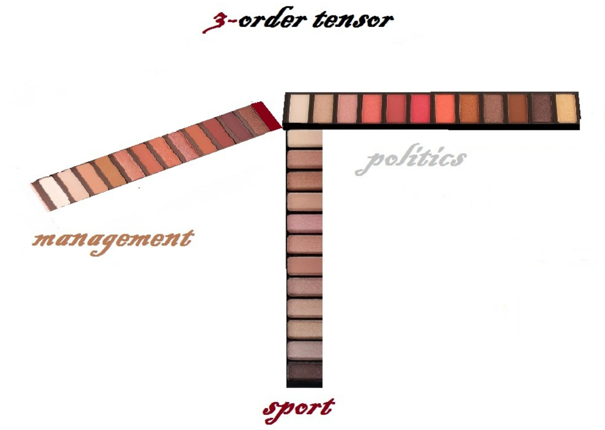
* We compute the 3-order tensors associated to all c vectors

$$M_{3c} = \frac{l}{3}(c \otimes c \otimes c + 2\sum_{i=1}^{n} c_i(e_i \otimes e_i \otimes e_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j(e_i \otimes e_i \otimes e_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j(e_i \otimes e_j \otimes e_i) -$$

and we compute the mean $M_3 \in \mathbb{R}^{n*n*n}$.

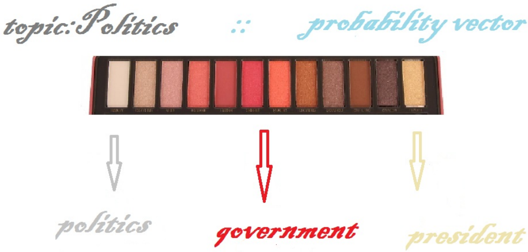
* We compute the singular value decomposition of $M_2 = USU^T$ and the whitening matrix $W = U_r S_r^{-\frac{1}{2}} \in \mathbb{R}^{n*r}$

* We compress the tensor M_3: $K_3 = (W^T, W^T, W^T).M_3$



∗ We decompose the tensor $K3 = \sum w_i u_i \otimes u_i \otimes u_i$ and deduce the decomposition $M3 = \sum w_i \mu_i \otimes \mu_i \otimes \mu_i$ where $\mu_i = W u_i$

∗ Each topic is represented by a probability vector and each component of it is equal to the probability of a word belongs to this topic ∗ We project a vector c corresponding to a fixed document on the new basis of vectors to compute the weights of each topic in the document.



Ex1:Decomposition of Real coefficients Multi-Linear Tensor

$\mathbf{x} = (x_0, x_1, x_2), \mathbf{y} = (y_0, y_1, y_2), \mathbf{z} = (z_0, z_1, z_2)$.
$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -0.5x_0y_0z_0 - 2.75x_0y_0z_1 - 0.75x_0y_1z_0 - 6.0x_0y_1z_1 + 0.375x_0y_1z_2 + 0.75x_0y_2z_0 + 2.25x_0y_2z_1 + 0.375x_0y_2z_2 + x_1y_0z_0 + 1.75x_1y_0z_1 + 0.75x_1y_0z_2 + 2.25x_1y_1z_0 + 3.0x_1y_1z_1 + 1.875x_1y_1z_2 - 0.75x_1y_2z_0 - 2.25x_1y_2z_1 - 0.375x_1y_2z_2 + 0.625x_2y_0z_0 - 1.25x_2y_0z_1 + 0.9375x_2y_0z_2 + 1.875x_2y_1z_0 - 3.75x_2y_1z_1 + 2.8125x_2y_1z_2$

$P(\mathbf{x}) = \sum_{i=1}^{r} \omega_i(\mathbf{a}_i(\mathbf{x}))(\mathbf{b}_i(\mathbf{y})(\mathbf{c}_i(\mathbf{z})) = \sum_{i=1}^{r} \omega_i(a_{i,0}x_0 + a_{i,1}x_1 + a_{i,2}x_2) * (b_{i,0}y_0 + b_{i,1}y_1 + b_{i,2}y_2) * (c_{i,0}z_0 + c_{i,1}z_1 + c_{i,2}z_2)$

$\mathbf{r} = 2$

Points =

$\mathbf{u}_i$	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	$b_{i,0}$	$b_{i,1}$	$b_{i,2}$	$c_{i,0}$	$c_{i,1}$	$c_{i,2}$
$\mathbf{u}_1$	1	-1.0	-0.0	1	2.0	-1.0	1	3	0.5
$\mathbf{u}_2$	1	1.0	2.5	1	3	$1.4 * 10^{-17}$	1	-2.0	1.5

Weights =

ω_1	-0.75
ω_2	0.25

Ex2:Decomposition of Integer coefficients Multi-Linear Tensor

$\mathbf{x} = (x_0, x_1, x_2, x_3), \mathbf{y} = (y_0, y_1, y_2, y_3), \mathbf{z} = (z_0, z_1, z_2, z_3)$.

$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -11.0x_1y_2z_3 - x_1y_3z_0 + x_1y_3z_2 - 3.0x_1y_3z_1 - 3.0x_1y_3z_3 - 7.0x_2y_0z_1 + 3.0x_2y_0z_0 + 3.0x_2y_0z_2 - 7.0x_2y_0z_3 + 7.0x_2y_1z_0 - 21.0x_2y_1z_1 + 17.0x_2y_1z_2 - 21.0x_2y_1z_3 + 3.0x_2y_2z_0 - 13.0x_2y_2z_1 + 9.0x_2y_2z_2 - 13.0x_2y_2z_3 - 5.0x_2y_3z_0 + 3.0x_2y_3z_1 - 7.0x_2y_3z_2 + 3.0x_0y_0z_0 - 6.0x_0y_0z_1 + 6.0x_0y_1z_0 + 4.0x_0y_0z_2 - 6.0x_0y_0z_3 + 12.0x_0y_1z_2 - 14.0x_0y_1z_1 + 16.0x_0y_2z_2 + 8.0x_0y_2z_0 - 14.0x_0y_1z_3 - 18.0x_0y_2z_1 - 18.0x_0y_2z_3 + 12.0x_0y_3z_0 + 24.0x_0y_3z_2 - 26.0x_0y_3z_1 + 2.0x_1y_0z_0 - 26.0x_0y_3z_3 + 3.0x_1y_0z_2 - 5.0x_1y_0z_1 - 5.0x_1y_0z_3 + 5.0x_1y_1z_0 - 15.0x_1y_1z_1 + 13.0x_1y_1z_2 - 15.0x_1y_1z_3 + 3.0x_1y_2z_0 - 11.0x_1y_2z_1 + 9.0x_1y_2z_2 + 3.0x_2y_3z_3 + 4.0x_3y_0z_0 + 9.0x_3y_2z_2 - 15.0x_3y_2z_1 + 3.0x_3y_2z_0 - 27.0x_3y_1z_3 + 21.0x_3y_1z_2 - 27.0x_3y_1z_1 + 9.0x_3y_1z_0 - 9.0x_3y_0z_3 + 3.0x_3y_0z_2 - 9.0x_3y_0z_1 + 9.0x_3y_3z_3 - 15.0x_3y_3z_2 + 9.0x_3y_3z_1 - 9.0x_3y_3z_0 - 15.0x_3y_2z_3$

$P(\mathbf{x}) = \sum_{i=1}^{r} \omega_i(\mathbf{a}_i(\mathbf{x}))(\mathbf{b}_i(\mathbf{y})(\mathbf{c}_i(\mathbf{z})) = \sum_{i=1}^{r} \omega_i(a_{i,0}x_0 + a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3) * (b_{i,0}y_0 + b_{i,1}y_1 + b_{i,2}y_2 + b_{i,3}y_3) * (c_{i,0}z_0 + c_{i,1}z_1 + c_{i,2}z_2 + c_{i,3}z_3)$

Points =

$\mathbf{u}_i$	$a_{i,1}$	$a_{i,2}$	$a_{i,3}$	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$c_{i,1}$	$c_{i,2}$	$c_{i,3}$
$\mathbf{u}_1$	-0.9	-1.9	-2.9	1.9	4	7.9	-1.9	1.9	-1.9
$\mathbf{u}_2$	1.9	2.9	3.9	2.9	2.9	2.9	-3	3	-3
$\mathbf{u}_3$	1	2	3	1	1	1	-1	-0.9	-1

with $a_{i,0} = b_{i,0} = c_{i,0} = 1$ for $i = 1, \ldots, r$ where $\mathbf{r} = 3$

Weights =

ω_1	ω_2	ω_3
1.0	0.9	0.9

Multi-symmetric tensor

$(E_j)_{1\leq j\leq k} | dim(E_j) = n_j + 1, E_j = \langle \mathbf{x}_j \rangle = \langle x_j, \dots, x_{j,n_j} \rangle$.

$\mathcal{S}^{\delta_j}(E_j) = \{p(\mathbf{x}_j)\text{homogeneous}, degree(p(\mathbf{x}_j)) = \delta_j\}$.

$\mathcal{S}^{\delta}(E) = \mathcal{S}^{\delta_1}(E_1) \otimes \mathcal{S}^{\delta_2}(E_2) \otimes \dots \otimes \mathcal{S}^{\delta_k}(E_k)$.
$[T] \in \mathcal{S}^{\delta}(E)$ is a multi symmetric tensor.

Notation: $\underline{\mathbf{x}} = (\underline{x}_1, \dots, \underline{x}_k)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ and
$\underline{x}_j = (x_{j,1}, \dots, x_{j,n_j})$ and $\alpha_j = (\alpha_{j,0}, \dots, \alpha_{j,n_j})$
so that $\underline{T}(\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k) = \sum_{\alpha_j \in \mathbb{N}^{n_j+1}, |\alpha_j| \leq \delta_j} t_{\alpha_1,\dots,\alpha_k} \underline{\mathbf{x}}^{\alpha}$ where
$\underline{\mathbf{x}}^{\alpha} = \prod_{j=1}^{k} \prod_{p=1}^{n_j} {x_{j,p}}^{\alpha_{j,p}}$

Multi-symmetric tensor

$T(\mathbf{x}) = T(\mathbf{x}_1, \ldots, \mathbf{x}_k)$: an multi-homogeneous polynomial of degree δ_j in the variable $\mathbf{x}_j = (x_{j,0}, \ldots, x_{j,n_j}) \rightsquigarrow$
$[T] = [t_{\alpha_1', \alpha_2', \ldots, \alpha_k'}]_{\substack{|\alpha_j'| = \delta_j \\ \alpha_j' \in \mathbb{N}^{n_j+1}}}$: multi symmetric array of coefficients such that each $\alpha_j' = (\alpha_{j,p_j}')_{0 \le p_j \le n_j}$ is a multi-index for $1 \le j \le k$.

For $|\alpha_j| \le \delta_j$, we denote $\bar{\alpha}_j := (\delta_j - |\alpha_j|, \alpha_{j,1}, \ldots, \alpha_{j,n_j}), j = 1, \ldots, k$
We identify $t_{\alpha_1', \ldots, \alpha_k'} := t_{\bar{\alpha}_1, \ldots, \bar{\alpha}_k}$.

Let $u_{i,j,0} \neq 0, j = 1, \ldots, k, i = 1, \ldots, r$,
then for $(u_{i,j,0})' = 1$ and $x_{j,0} = 1 \rightsquigarrow$

- $R = \mathbb{C}[\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k]$ where $\underline{\mathbf{x}}_j = (x_{j,1}, \ldots, x_{j,n_j})$ for $j = 1, \ldots, k$
- $R_{\delta_1, \delta_2, \ldots, \delta_k} = \{T \in \mathcal{S}^{\delta}(E) | x_{j,0} = 1, j = 1, \ldots, k\}$

Multi Symmetric Tensor Decomposition

Sum of products of power of linear forms:

$T(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \sum_{p=1}^{r} \omega_p \mathbf{u}_{p,1}^{\delta_1}(\mathbf{x}_1)\mathbf{u}_{p,2}^{\delta_2}(\mathbf{x}_2)\ldots\mathbf{u}_{i,k}^{\delta_k}(\mathbf{x}_k)$ where $\mathbf{u}_{p,j}(\mathbf{x}_j) = u_{p,j}x_j + u_{p,j,1}x_{j,1} + \ldots + u_{p,j,n_j}x_{j,n_j}$ and $\mathbf{u}_p = (u_{p,j,p_j})_{\substack{0\le p_j\le n_j \\ 1\le j\le k}} \in \mathbb{C}^{\sum_{j=1}^{k}(n_j+1)}$ is the coefficient vector associated to the linear forms $\mathbf{u}_{p,j}(\mathbf{x}_j)$ in the basis $\mathbf{x}_j$ for $j = 1, \ldots, k$.

Rank r of T: Minimal number of terms in a decomposition of $T(\mathbf{x})$.

By a generic change of coordinates in each E_j, we may assume that $u_{p,j} \neq 0$ and that T has an affine decomposition. Then by scaling $\mathbf{u}_p(\mathbf{x})$ and multiplying ω_p by the $d^{\mathbf{th}}$ power of the scaling factor we may assume that $u_{p,j} = 1$ for $p = 1, \ldots, r$ and $j = 1, \ldots, k$. Thus the polynomial $\underline{T}(\underline{\mathbf{x}}) = \sum_{p=1}^{r} \omega'_p \mathbf{u}'^{\,\delta}_p(\underline{\mathbf{x}}) = \sum_{p=1}^{r} \omega'_i \mathbf{u}'^{\,\delta_1}_{p,1}(\underline{\mathbf{x}}_1)\mathbf{u}'^{\,\delta_2}_{p,2}(\underline{\mathbf{x}}_2)\ldots\mathbf{u}'^{\,\delta_k}_{p,k}(\underline{\mathbf{x}}_k)$

$T_1(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k),\ T_2(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \in \mathcal{S}^\delta(E) \rightsquigarrow$

Apolar Product:

$$\langle \underline{T_1}(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k), \underline{T_2}(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k)\rangle = \sum_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \tau^{(1)}_{\alpha_1, \alpha_2, \ldots, \alpha_k} \bar{\tau}^{(2)}_{\alpha_1, \alpha_2, \ldots, \alpha_k} \binom{\delta}{\alpha}$$

where $\binom{\delta}{\alpha} = \binom{\delta_1}{\alpha_1}\binom{\delta_2}{\alpha_2}\cdots\binom{\delta_k}{\alpha^k}$.

Dual Operator:

$$T^* : (R_{\delta_1, \delta_2, \ldots, \delta_k}) \to (R_{\delta_1, \delta_2, \ldots, \delta_k})^* \quad (1)$$

$$\underline{T_2} \mapsto T^*(\underline{T_2}) = \langle \underline{T}(\underline{\mathbf{x}}), \underline{T_2}(\underline{\mathbf{x}})\rangle \quad (2)$$

For $T = (t_{\alpha_1, \alpha_2, \ldots, \alpha_k})_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \in \mathcal{S}^\delta(E) \rightsquigarrow$

- $\tau_{\alpha_1, \alpha_2, \ldots, \alpha_k}(T) = \tau_{\alpha_1, \alpha_2, \ldots, \alpha_k} = t_{\alpha_1, \alpha_2, \ldots, \alpha_k} \binom{\delta_1}{\alpha_1}^{-1}\binom{\delta_2}{\alpha_2}^{-1}\cdots\binom{\delta_k}{\alpha_k}^{-1}$.
- Dual via the formal power series:

$$\tau(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k) = T^*(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k) = \sum_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \tau_{\alpha_1, \alpha_2, \ldots, \alpha_k} \frac{(\mathbf{y}_1)^{\bar{\alpha}_1}}{\bar{\alpha}_1!} \frac{(\mathbf{y}_2)^{\bar{\alpha}_2}}{\bar{\alpha}_2!} \cdots \frac{(\mathbf{y}_k)^{\bar{\alpha}_k}}{\bar{\alpha}_k!}$$

where $(\mathbf{y}_j)^{\bar{\alpha}_j} = (y_j, y_{j,1}, \ldots, y_{j,n_j})^{(\alpha_j, \alpha_{j,1}, \ldots, \alpha_{j,n_j})} = \prod_{p_j=0}^{n_j} (y_{j,p_j})^{\alpha_{j,p_j}}$ for $j = 1, \ldots, k$

Dual of $\mathbf{u}_1^{\delta_1}\mathbf{u}_2^{\delta_2}\ldots{\mathbf{u}_k}^{\delta_k}$ is the evaluation $\mathbf{e}_{\mathbf{u}}$ at $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$.

Thus if $T = \sum_{i=1}^{r} \omega_i \mathbf{u}_{i,1}^{\delta_1}\mathbf{u}_{i,2}^{\delta_2}\ldots{\mathbf{u}_{i,k}}^{\delta_k}$, then T^* coincides with the weighted sum of evaluations $T^* = \sum_i^r \omega_i\, \mathbf{e}_{\mathbf{u}_i}$ on $R_{\delta_1,\delta_2,\ldots,\delta_k}$.

Dual space

Power Formal series :

$$\tau(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \tau_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{C}[[\mathbf{y}]]$$

Linear Functional :

$$\begin{array}{rcl} \tau : \mathbb{C}[\mathbf{x}] & \to & \mathbb{C} \\ p = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha & \mapsto & \langle \tau \mid p \rangle = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \tau_\alpha . \end{array}$$

Duality :

$$\mathbb{C}^{\mathbb{N}^n} \equiv \mathbb{C}[\mathbf{x}]^* \equiv \mathbb{C}[[\mathbf{y}]]$$

$$L_0(\mathbb{C}^{\mathbb{N}^n}) \equiv \mathbb{C}[\mathbf{x}]$$

$\rightsquigarrow$ *Hankel Operator* :

$$\begin{array}{rcl} H_\tau : \mathbb{C}[\mathbf{x}] & \to & \mathbb{C}[[\mathbf{y}]] \\ p & \mapsto & p \star \tau \end{array}$$

Hankel Matrices

Hankel Matrices:

$$H = [\tau_{i+j}]_{0 \le i \le l, 0 \le j \le m}$$

Multivariate Hankel Matrices:

$$H = [\tau_{\alpha+\beta}]_{\alpha \in A, \beta \in B}$$

Multivariate Hankel operators: $\tau = (\tau_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}^{\mathbb{N}^n} \rightsquigarrow$

$$\begin{array}{rcl} H_\tau : L_0(\mathbb{C}^{\mathbb{N}^n}) & \to & \mathbb{C}^{\mathbb{N}^n} \\ (p_\alpha)_\alpha & \mapsto & (\sum_\alpha p_\alpha \tau_{\alpha+\beta})_{\beta \in \mathbb{N}^n} \end{array} \quad (3)$$

Goal:Decomposition of Hankel operator

The moment matrix $H_\tau^{A,B}$ of τ: $(\mathbf{x}^\beta)_{\beta \in B}$ and $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha \in A} \rightsquigarrow$

$$H_\tau^{A,B} = [\tau_{\alpha+\beta}]_{\alpha \in A, \beta \in B}.$$

The evaluation at ξ:

$\mathbf{e}_\xi(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \xi^\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = e^{\mathbf{y} \cdot \xi}$
$\rightsquigarrow \forall p \in R, < \mathbf{e}_\xi | p >= \sum_{\alpha \in \mathbb{N}^n} p_\alpha \, \xi^\alpha = p(\xi).$

- $H_{\mathbf{e}_\xi} : p \mapsto p \star \mathbf{e}_\xi = p(\xi)\mathbf{e}_\xi,$
- $H_\xi{}^{A,B} = [\xi^{\beta+\alpha}]_{\beta \in B, \alpha \in A}$, if $H_\xi{}^{A,B} \neq 0$

Decomposition of Hankel matrices

Decomposition of Quotient Algebra

Theo: I_τ kernel of H_τ :: $\mathcal{A}_\tau = \mathbb{C}[\mathbf{x}]/I_\tau$ gorenstein algebra $\rightsquigarrow$

- $\tau = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$ with $\omega_i \in \mathbb{C}[\mathbf{y}]$, $\xi_i \in \mathbb{C}^n$ where ξ_i are distinct and of multiplicity one.
- H_τ is of rank r,
- $\mathcal{A}_\tau$ is an Artinian Gorenstein algebra of dimension r.
- $\mathcal{V}(I) = \{\xi_1, \ldots, \xi_r\}$

Rq The decomposition problem τ as a weighted sum of products of power of linear forms reduces to the solution of the polynomial equations $p = 0$ for p in the kernel I_τ of H_τ.

$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I_\tau$ is Artinian :: finite dimension over $\mathbb{C}$

$\rightsquigarrow$ I_τ and $\mathcal{V}(I_\tau) = \{\xi_1, \ldots, \xi_r\} = \{\xi \in \mathbb{C}^n \mid \forall p \in I_\tau, p(\xi) = 0\}$

$\rightsquigarrow$ decomposition of $\mathcal{A}$ as a sum of sub-algebras:

$$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I_\tau = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$$

where $\mathcal{A}_i = \mathbf{u}_{\xi_i}\mathcal{A} \sim \mathbb{C}[\mathbf{x}]/Q_i$ and Q_i is the primary component of I associated to the root $\xi_i \in \mathbb{C}^n$.

The idempotents $\mathbf{u}_{\xi_1}, \ldots, \mathbf{u}_{\xi_r}$ :: $\mathbf{u}_{\xi_i}^2(\mathbf{x}) \equiv \mathbf{u}_{\xi_i}(\mathbf{x})$, $\sum_{i=1}^{r} \mathbf{u}_{\xi_i}(\mathbf{x}) \equiv 1$.

Decomposition of Hankel matrices

Multiplication Operator

Multiplication Operator: $g \in \mathbb{C}[\mathbf{x}]$, $\mathcal{M}_g$

$$\begin{array}{rccc} \mathcal{M}_g : & \mathcal{A} & \rightarrow & \mathcal{A} \\ & h & \mapsto & \mathcal{M}_g(h) = g\, h. \end{array}$$

Transpose of Multiplication Operator:

$$\begin{array}{rccc} \mathcal{M}_g{}^{\mathsf{T}} : & \mathcal{A}^* & \rightarrow & \mathcal{A}^* \\ & \Lambda & \mapsto & \mathcal{M}_g{}^{\mathsf{T}}(\Lambda) = \Lambda \circ \mathcal{M}_g = g \star \Lambda. \end{array}$$

Decomposition of Hankel matrices

Eigenvalues and eigenvectors

Let I ideal of $\mathbb{C}[\mathbf{x}]$ and $\mathcal{V}(I) = \{\xi_1, \xi_2, \ldots, \xi_r\}$ such that ξ_i are simple $\Rightarrow$

- $\forall g \in \mathcal{A}$, the eigenvalues of $\mathcal{M}_g$ and $\mathcal{M}_g{}^{\mathsf{T}}$ are the values $g(\xi_1), \ldots, g(\xi_r)$ of the polynomial g at the roots with multiplicities $\mu_i = \dim \mathcal{A}_i = 1$.
- The eigenvectors common to all $\mathcal{M}_g{}^{\mathsf{T}}$ with $g \in \mathcal{A}$ are - up to a scalar - the evaluations $\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r}$.
- If g is separating the roots, i.e. $g(\xi_p) \neq g(\xi_q)$ for $p \neq q$, then the eigenvectors of $\mathcal{M}_g$ are, up to a scalar, interpolation polynomials $\mathbf{u}_{\xi_i}$ at the roots ξ_i.

Decomposition of Hankel matrices

Bases

Lemma:Let $B = \{b_1, \ldots, b_r\}$, $B' = \{b'_1, \ldots, b_r{}'\} \subset \mathbb{C}[\mathbf{x}]$. If the matrix $H_\tau^{B,B'} = (\langle \tau | b_i b'_j \rangle)_{1 \le i,j \le r}$ is invertible $\Rightarrow$ B and B' are linearly independent in $\mathcal{A}_\tau$.

Decomposition of Hankel matrices

Multiplication operator via truncated Hankel Matrices

Prop: Let B, B' be basis of $\mathcal{A}_\tau$ and $g \in \mathbb{C}[\mathbf{x}]$. We have

$$H_{g\star\tau}^{B,B'} = ({M_g}^{B})^{\mathsf{T}} H_{\tau}^{B,B'} = H_{\tau}^{B,B'} {M_g}^{B'}. \quad (4)$$

where ${M_g}^{B}$ (resp. ${M_g}^{B'}$) is the matrix of the multiplication by g in the basis B (resp. B') of $\mathcal{A}_\tau$.

Let $\tau(\mathbf{y}) = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$ with $\omega_i \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct and simple.
Let B, B' be bases of $\mathcal{A}_\tau$ $\rightsquigarrow$

- For $g \in \mathbb{C}[\mathbf{x}]$, ${M_g}^{B'} = (H_{\tau}^{B,B'})^{-1} H_{g\star\tau}^{B,B'}$, $({M_g}^{B})^{\mathsf{T}} = H_{g\star\tau}^{B,B'} (H_{\tau}^{B,B'})^{-1}$.
- For $g \in \mathbb{C}[\mathbf{x}]$, the generalized eigenvalues of $(H_{g\star\tau}^{B,B'}, H_{\tau}^{B,B'})$ are $g(\xi_i)$ with multiplicity 1, $i = 1, \ldots, r$.
- The generalized eigenvectors common to all $(H_{g\star\tau}^{B,B'}, H_{\tau}^{B,B'})$ for $g \in \mathbb{C}[\mathbf{x}]$ are - up to a scalar - $(H_{\tau}^{B,B'})^{-1} B(\xi_i)$, $i = 1, \ldots, r$.

Multi-Linear Tensor Decomposition Problem

$A_1, A_2 \mid R_{1,1,\ldots,1} = \langle A_1 * A_2 x_{j,i_j}, 0 \leq i_j \leq n_j \rangle$

- $\widehat{H}_{T^*}^{A_1,A_2} = [t_{\alpha+\beta}]_{\alpha \in A_1 \beta \in A_2}$ and $\widehat{H}_{T^*}^{\overline{A}_1,\overline{A}_2}$
- $\widehat{H}_0 = \widehat{H}_{T^*}^{\overline{A}_1,\overline{A}_2}$
- $\widehat{H}_{2,i_2} = \widehat{H}_{x_{2,i_2} \star T^*}^{\overline{A}_1,\overline{A}_2} = \widehat{H}_{T^*}^{x_{2,i_2} \star \overline{A}_1,\overline{A}_2} = [t_{\alpha+\beta}]_{\alpha \in x_{2,i_2} \star \overline{A}_1, \beta \in \overline{A}_2}$
- $\widehat{H}_0 = \sum_{i_1=0}^{n_1} \lambda_{i_1} \widehat{H}_{1,i_1}$

Multi-Linear Tensor Decomposition Problem

Singular Value Decomposition

Singular Value Decomposition:

$$\widehat{H}_0 = USV^{\mathsf{T}}$$

- Vectors: $u_i = [u_{\alpha,i}]_{\alpha \in \overline{A}_1}$, $v_j = [v_{\beta,j}]_{\beta \in \overline{A}_2}$ :: i^{th} and j^{th} col of U^{H} and $\overline{V}$
- Polynomials: $u_i(\mathbf{x}) = u_i^{\mathsf{T}}\, \mathbf{x}^{\overline{A}_1} = \sum_{\alpha \in \overline{A}_1} u_{\alpha,i} \mathbf{x}^{\alpha}$ and $v_j(\mathbf{x}) = v_j^{\mathsf{T}}\, \mathbf{x}^{\overline{A}_2} = \sum_{\beta \in \overline{A}_2} v_{\beta,j} \mathbf{x}^{\beta}$.
- Bases: $U_r{}^{\mathsf{H}} := (u_i(\mathbf{x}))_{i=1,\dots,r}$ and $\overline{V}_r := (v_j(\mathbf{x}))_{j=1,\dots,r}$

- Truncated Singular Value Decomposition: $\widehat{H}_0$,$\widehat{H}_{x_{2,i_2}}$ and $\widehat{H}_0 \rightsquigarrow \widehat{H}^r_0$,$\widehat{H}^r_{x_{2,i_2}}$ and $\widehat{H}^r_0$
- $$\widehat{H}^r_{x_{2,i_2}} = (M^{{U_r}^{\mathbf{H}}}_{x_{2,i_2}})^{\mathsf{T}} \widehat{H}^r_0 = \widehat{H}^r_0 M^{\overline{V}_r}_{x_{2,i_2}}$$
  where $M^{{U_r}^{\mathbf{H}}}_{x_{2,i_2}}$ (resp. $M^{\overline{V}_r}_{x_{2,i_2}}$) is the multiplication matrix by x_{2,i_2} in the basis ${U_r}^{\mathbf{H}}$ (resp. $\overline{V}_r$) and $M^{\overline{V}_r}_{x_{2,i_2}}$ is the multiplication matrix by x_{2,i_2} in the basis $\overline{V}_r$.
- By linearity: $\widehat{H}^r_0 = \sum_{i_1=0}^{n_2} \lambda_{i_2} \widehat{H}^r_{x_{2,i_2}} = \widehat{H}^r_0 \sum_{i_1=0}^{n_2} \lambda_{i_2} M^{\overline{V}_r}_{x_{2,i_2}} = \widehat{H}^r_0 M^{\overline{V}_r}_{\lambda(x_2)} \Rightarrow$
  $$(\widehat{H}^r_0)^{-1} = (M^{\overline{V}_r}_{\lambda(x_2)})^{-1} (\widehat{H}^r_0)^{-1}$$
  $$(\widehat{H}^r_0)^{-1} \widehat{H}^r_{x_{2,i_2}} = (M^{\overline{V}_r}_{\lambda(x_2)})^{-1} M^{\overline{V}_r}_{x_{2,i_2}} = M^{\overline{V}_r}_{(x_{2,i_2}/\lambda(x_2))}$$

We compute the eigenvalues and the eigenvectors of the multiplication matrices $M^{\overline{V}_r}_{(x_{2,i_2}/\lambda(x_2))}$ in order to obtain the weights and the points of the decomposition.

Multiplication Operators in the orthogonal basis

Prop: Let $\tau = \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\xi_i}$ with $\omega_i \in \mathbb{C}$, $\xi_i \in \mathbb{C}^n$ are simple.
If rank $H_{\tau}^{\overline{A}_1, \overline{A}_2} = r$,

- the sets of polynomials $U_r{}^{\mathsf{H}}$ and $\overline{V}_r$ are bases of $\mathcal{A}_\tau$.
- The matrix $M_{x_{2,i_2}}^{\overline{V}_r}$ associated to the multiplication operator by y_i in the basis $\overline{V}_r$ of $\mathcal{A}_\tau$ is $M_{x_{2,i_2}}^{\overline{V}_r} = S_r{}^{-1} U_r{}^{\mathsf{H}} H_{x_{2,i_2}\star\tau}^{\overline{A}_1, \overline{A}_2} \overline{V}_r$ $i = 1, \ldots, n$.

Algorithm 5.1: Decomposition of Tri-Linear Tensor with constant weights

the moments $(t_{i,j,k})_{\substack{0\le i\le n_1\\0\le j\le n_2\\0\le k\le n_3}}$ of τ.

1. Compute the monomial sets $A_1 = (x_i)_{0\le i\le n_1}$ and $A_2 = (z_k)_{0\le k\le n_3}$ and substitute the x_0, y_0 and z_0 by 1 to define $\overline{A}_1$ and $\overline{A}_2$.
2. Compute the Hankel matrix $H_\tau^{A_1,A_2} = [t_{\alpha+\beta}]_{\alpha\in A_1\beta\in A_2}$ and deduce the truncated Hankel matrix $H_\tau^{\overline{A}_1,\overline{A}_2}$ for the monomial sets $\overline{A}_1$ and $\overline{A}_2$.
3. Compute the singular value decomposition of $H_\tau^{\overline{A}_1,\overline{A}_2} = USV^{\mathsf{T}}$ where $\overline{A}_1 = \langle 1, x_1, \ldots, x_{n_1}\rangle$ and $\overline{A}_2 = \langle 1, z_1, \ldots, z_{n_3}\rangle$ with singular values $s_1 \ge s_2 \ge \cdots \ge s_r \ge 0$.
4. Determine its numerical rank, that is, the largest integer r such that $\frac{s_r}{s_1} \ge \epsilon$.
5. Form the multiplication matrices by x_{2,j_2} in the basis $\overline{V}_r$, $M_{x_{2,j_2}}^{\overline{V}_r} = S_r^{-1} U_r^{\mathsf{H}} H_{x_{2,j_2}\star\tau}^{\overline{A}_1,\overline{A}_2} \overline{V}_r$ where $H_{x_{2,i_2}\star\tau}^{\overline{A}_1,\overline{A}_2}$ is the Hankel matrix associated to $x_{2,j_2}\star\tau$ for $j = 1, \ldots, n_2$.

6. Compute the eigenvectors $\mathbf{v}_p$ of $\sum_{j=1}^{n_2} l_j M_{x_{2,j_2}}^{\overline{V}_r}$ such that $|l_j| \leq 1, j = 1, \ldots, n_2$ and for each $p = 1, \ldots, r$ do the following:

- The $y's$ coordinates of the $\mathbf{u}_p$ are the eigenvalues of the multiplication matrices by x_{2,j_2}. Use the formula $M_{x_{2,j_2}}^{\overline{V}_r} \mathbf{v}_p = b_{p,j} \mathbf{v}_p$ for $p = 1, \ldots, r$ and $j = 1, \ldots, n_2$ and deduce the $b_{p,j}$.
- Write the matrix $H_\tau^{\overline{A}_1, \overline{A}_2}$ in the basis of interpolation polynomials(ie. the eigenvectors $\mathbf{v}_p$) and use the corresponding matrix $\mathcal{T} = [\tau(x_{3,i_3} \mathbf{v}_j)]_{\substack{1 \leq i \leq n_3 \\ 1 \leq j \leq r}}$ to compute the $z's$ coordinates. Divide the $k^{\mathbf{th}}$ row on the first row of the matrix $\mathcal{T}$ to obtain the values of $c_{p,k}$ for $p = 1, \ldots, r$ and $k = 1, \ldots, n_3$.
- The $x's$ coordinates of $\underline{u}_p$ are computed using the eigenvectors of the transpose of the matrix $M_{x_{2,j_2}}^{\overline{V}_r}$. They -are up to scalar- the evaluations, they are represented by vectors of the form $\mathbf{v}_p^* = \mu_p [1, a_{p,1}, \ldots, a_{p,n_1}]$. Compute $\mathbf{v}_p^*$ as the p^{th} column of the transpose of the inverse of the matrix $V = [v_1, \ldots, v_r]$ for $p = 1, \ldots, r$ and deduce $a_{p,i} = \frac{\mathbf{v}_p^*[i+1]}{\mathbf{v}_p^*[1]}$ for $p = 1, \ldots, r$ and $i = 1, \ldots, n_1$.
- Compute $\omega_p = \frac{\langle \tau | \mathbf{v}_p \rangle}{\mathbf{v}_p(\mathbf{u}_p)}$.

Ex3:Decomposition of Real coefficients Tensor

$\mathbf{x} = (x_0, x_1, x_2, x_3)$, $\mathbf{y} = (y_0, y_1, y_2, y_3)$, $\mathbf{x} = (z_0, z_1, z_2, z_3)$.

$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0.02646466399x_0y_0z_0 - 0.02079158604x_0y_0z_1 - 0.006500462094x_0y_0z_2 - 0.02089539048x_0y_0z_3 - 0.05126123958x_0y_1z_0 + 0.03152394184x_0y_1z_1 - 0.03531344678x_0y_1z_2 + .1331916336x_0y_1z_3 + 0.00568937625x_0y_2z_0 - 0.01946921548x_0y_2z_1 + 0.02562423711x_0y_2z_2 - .1072261978x_0y_2z_3 - 0.02499583734x_0y_3z_0 + 0.02311010129x_0y_3z_1 + 0.007139677259x_0y_3z_2 + 0.02612360782x_0y_3z_3 - 0.001785167174x_1y_0z_0 - 0.01969431998x_1y_0z_1 - 0.01736839108x_1y_0z_2 - 0.009273636923x_1y_0z_3 + .1287214315x_1y_1z_0 + 0.02535497757x_1y_1z_1 + 0.04009363745x_1y_1z_2 + 0.04644350153x_1y_1z_3 - 0.01491990412x_1y_2z_0 + 0.02842163781x_1y_2z_1 + 0.01170275867x_1y_2z_2 + 0.03476034515x_1y_2z_3 - 0.01019076172x_1y_3z_0 + 0.01466986899x_1y_3z_1 + 0.01312158715x_1y_3z_2 - 0.000552569691x_1y_3z_3 - 0.01996141762x_2y_0z_0 + 0.02021382968x_2y_0z_1 + 0.002520024556x_2y_0z_2 + 0.03293876384x_2y_0z_3 + 0.05285608034x_2y_1z_0 - 0.06615741785x_2y_1z_1 + 0.03526018401x_2y_1z_2 - .2197159500x_2y_1z_3 - 0.02684819787x_2y_2z_0 + 0.04111686744x_2y_2z_1 - 0.02889572651x_2y_2z_2 + .1569483632x_2y_2z_3 + 0.02095884678x_2y_3z_0 - 0.02208293066x_2y_3z_1 - 0.002096727077x_2y_3z_2 - 0.03807484757x_2y_3z_3 + 0.02671616684x_3y_0z_0 - 0.02253751391x_3y_0z_1 - 0.01818839314x_3y_0z_2 + 0.002862651160x_3y_0z_3 + 0.02576052725x_3y_1z_0 - 0.02679549338x_3y_1z_1 - 0.005080088212x_3y_1z_2 - 0.03991979132x_3y_1z_3 - 0.03800977672x_3y_2z_0 + 0.03595197122x_3y_2z_1 + 0.01574387152x_3y_2z_2 + 0.03005675208x_3y_2z_3 - 0.02684599409x_3y_3z_0 + 0.02258264356x_3y_3z_1 + 0.01852935177x_3y_3z_2 - 0.003645021223x_3y_3z_3$

$$H_{\tau}^{\overline{A}_1,\overline{A}_2} = USV^{T} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} 1 & x_1 & x_2 & x_3 \end{array}$$

$$\begin{bmatrix} t_{0,0,0} & t_{1,0,0} & t_{2,0,0} & t_{3,0,0} \\ t_{0,0,1} & t_{1,0,1} & t_{2,0,1} & t_{3,0,1} \\ t_{0,0,2} & t_{1,0,2} & t_{2,0,2} & t_{3,0,2} \\ t_{0,0,3} & t_{1,0,3} & t_{2,0,3} & t_{3,0,3} \end{bmatrix} \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array}$$

$$= \begin{array}{cccc} 1 & x_1 & x_2 & x_3 \end{array}$$

$$\begin{bmatrix} 0.02646466 & -0.001785167 & -0.01996141 & 0.02671616 \\ -0.02079158 & 0.01969431 & 0.02021382 & -0.02253751 \\ -0.006500462 & -0.01736839 & 0.002520024 & -0.01818839 \\ -0.02089539 & -0.009273636 & 0.03293876 & 0.002862651 \end{bmatrix} \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array}$$

$$S = \begin{bmatrix} 0.0681423308704 & 0 & 0 & 0 \\ 0 & 0.0284093200680 & 0 & 0 \\ 0 & 0 & 0.0199716008591 & 0 \\ 0 & 0 & 0 & 3.3112 * 10^{-12} \end{bmatrix}$$

epsilon$= 10^{-10}$ **r**$= 3$

$$H_{y_i \tau}^{\overline{A}_1, \overline{A}_2} = \begin{array}{c} \begin{array}{cccc} y_i 1 & y_i x_1 & y_i x_2 & y_i x_3 \end{array} \\ \begin{bmatrix} t_{0,0,0} & t_{1,0,0} & t_{2,0,0} & t_{3,0,0} \\ t_{0,0,1} & t_{1,0,1} & t_{2,0,1} & t_{3,0,1} \\ t_{0,0,2} & t_{1,0,2} & t_{2,0,2} & t_{3,0,2} \\ t_{0,0,3} & t_{1,0,3} & t_{2,0,3} & t_{3,0,3} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array}$$

for $i = 1, \ldots, n$

$$H_{y_1\tau}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} -0.05126123 & 0.1287214 & ,0.05285608 & 0.02576052 \\ 0.03152394 & 0.02535497 & -0.06615741 & -0.02679549 \\ -0.03531344 & 0.04009363 & 0.03526018 & -0.005080088 \\ 0.1331916 & 0.04644350 & -0.2197159 & -0.03991979 \end{bmatrix}$$

$$H_{y_2\tau}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} 0.00568937 & -0.01491990 & -0.02684819 & -0.03800977 \\ -0.01946921 & 0.02842163 & 0.04111686 & 0.03595197 \\ 0.02562423 & 0.01170275 & -0.02889572 & 0.01574387 \\ -0.1072261 & 0.03476034 & 0.1569483 & 0.03005675 \end{bmatrix}$$

$$H_{y_3\tau}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} -0.02499583 & -0.01019076 & 0.02095884 & -0.02684599 \\ 0.02311010 & 0.01466986 & -0.02208293 & 0.02258264 \\ 0.007139677 & 0.01312158 & -0.002096727 & 0.01852935 \\ 0.02612360 & -0.000552569 & -0.03807484 & -0.003645021 \end{bmatrix}$$

Points =

$\mathbf{u}_i$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$
$a_{i,0}$	1	1	1
$a_{i,1}$	0.953586012938	-0.2373776281105	-4.7475616265100
$a_{i,2}$	-0.65267063880	-1.4691283332279	-0.14517159542469
$a_{i,3}$	1.6178837601907	-0.2519515268291	0.00749395381230
$b_{i,0}$	1	1	1
$b_{i,1}$	0.5811022172775	-6.91176612232	-6.463589779792
$b_{i,2}$	-1.110402543980	4.89705156939	-0.5801680510141
$b_{i,3}$	-1.0127399256801	-1.1502422172407	-0.29739057249700
$c_{i,0}$	1	1	1
$c_{i,1}$	-0.867731473730	-1.3065248943787	0.4138185554696
$c_{i,2}$	-0.6216530357344	0.5333491862721	0.37476316184358
$c_{i,3}$	-0.0957344227024	-3.9750361756013	0.757692579173

Weights =

ω_1	0.0173563883486
ω_2	0.00552219678715
ω_3	0.00358607886042