Rational minimax approximation via adaptive barycentric representations

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joint work with Bernhard Beckermann, Yuji Nakatsukasa
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Why are they important?

$\rightarrow$ powerful **approximations** near singularities or on unbounded domains

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Some applications:

- elementary + special functions
- recursive filter design
- matrix exponentials & stiff PDEs
- optimal control problems
- ...

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Output: $r^* \in \mathcal{R}_{m,n} = \left\{ \frac{p}{q}, p \in \mathbb{R}_m[x], q \in \mathbb{R}_n[x] \right\}$ s.t.

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- **Alternation Theorem** [Achieser 1930]:
  $\rightarrow f - r^*$ equioscillates at least $m + n + 2 - d$ times

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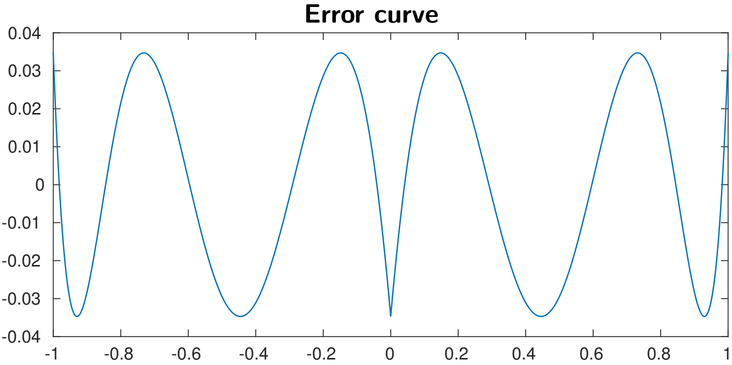

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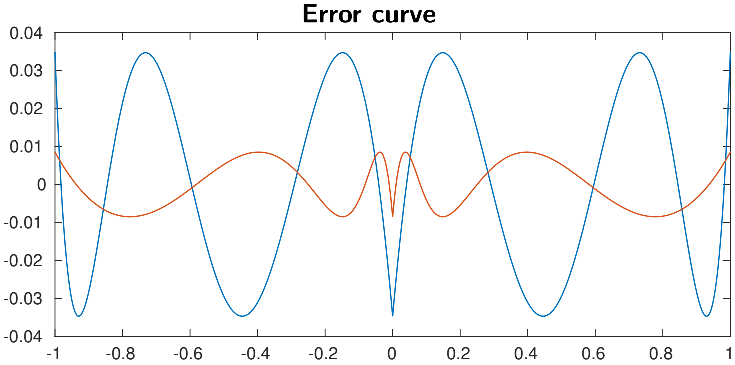

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$E_{n,0}(f) \sim \beta/n,$	$\beta = 0.2801...$	[Varga & Carpenter 1985]
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- Chebfun (Matlab): `minimax`

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$\rightarrow$ barycentric form for type (n, n) rational functions

$$r(z) = \frac{N(z)}{D(z)} = \sum_{k=0}^{n} \frac{\alpha_k}{z - t_k} \Bigg/ \sum_{k=0}^{n} \frac{\beta_k}{z - t_k}$$

Notation:

- $\{\alpha_k\}, \{\beta_k\}$ barycentric coefficients
- $\{t_k\}$ support points

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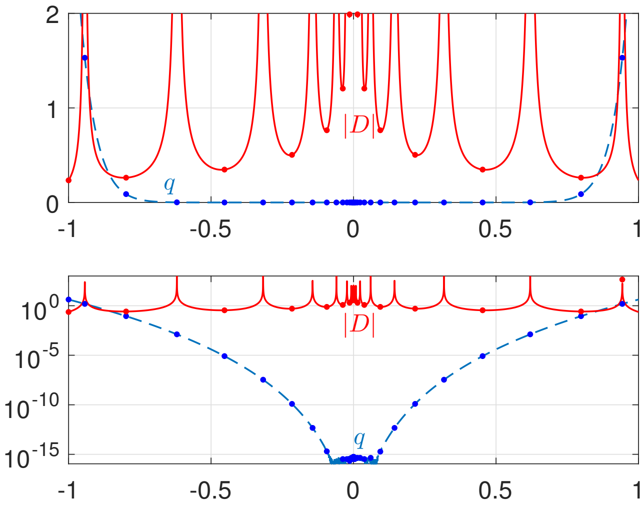
Example:

$\rightarrow$ the adaptive Antoulas-Anderson (AAA) algorithm [Nakatsukasa, Sète & Trefethen 2018]: greedy least squares approximation

Example: $f(x) = |x|, x \in [-1, 1]$, type $(20, 20)$

p/q vs N/D



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$$f(x_k) - r(x_k) = (-1)^{k+1}\lambda, \qquad k = 0, \ldots, 2n+1$$

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Step 3: among local extrema of $f - r$, take $2n + 2$ new points

$$a \leqslant x_0' < \cdots < x_{2n+1}' \leqslant b,$$

$f - r$ alternates in sign + at least one global extrema over $[a, b]$ and

$$|f(x_k') - r(x_k')| \geqslant |\lambda|, \qquad k = 0, \ldots, 2n+1$$

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Convergence:

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What can go wrong?

$\rightarrow$ no pole-free solution in **Step 2**

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- extrapolation from lower degree approx. ($(2,2),(3,3),(4,4),\ldots$)

Step 2: find r

$\rightarrow$ find $r = N/D \in \mathcal{R}_{n,n}$ s.t.

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→ matrix form

$$C\alpha = \left(\begin{bmatrix} f(x_0) & & & \\ & f(x_1) & & \\ & & \ddots & \\ & & & f(x_{2n+1}) \end{bmatrix} - \lambda \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{bmatrix} \right) C\beta,$$

$C \in \mathbb{R}^{(2n+2)\times(n+1)}$ Cauchy matrix, $C_{k,j} = 1/(x_k - t_j)$

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$$\begin{bmatrix} C & -FC \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} 0 & -SC \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$F = \mathrm{diag}(f(x_k)), S = \mathrm{diag}((-1)^{k+1})$

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$$Q_1^T (SF) Q_1 R\beta = \lambda R\beta,$$

where $\omega_x(x) = \prod_{k=0}^{2n+1}(x - x_k), \quad \omega_t(x) = \prod_{j=0}^{n}(x - t_j),$

$$\Delta = \text{diag}\left(\frac{\omega_t(x_0)^2}{\omega_x'(x_0)}, \ldots, \frac{\omega_t(x_{2n+1})^2}{\omega_x'(x_{2n+1})}\right)$$

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$\rightarrow$ we show that this happens (with optimum 1) for

$$t_k = x_{2k+1}, \qquad k = 0, \ldots, n$$

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$\rightarrow$ Chebyshev interpolants of $e(x) = f(x) - r(x)$ on each subinterval

$\rightarrow$ colleague matrix root finding [Specht, Good]

Examples

DEMO

Conclusion

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→ the details:
B. Beckermann, S.-I. Filip, Y. Nakatsukasa, L. N. Trefethen, *Rational minimax approximation via adaptive barycentric representations*, arXiv:1705.10132, under minor revision for *SIAM Journal on Scientific Computing*

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