Équations d'évolution et calcul différentiel non commutatifs.

Non-commutative differential equations
and systems of coordinates on (some) infinite dimensional Lie Groups.

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Collaboration at various stages of the work
and in the framework of the Project
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Foreword: Goal of this talk

In this talk, I will show tools and, if time permits, sketch proofs about Noncommutative Evolution Equations.

The main item of data is that of Noncommutative Formal Power Series with variable coefficients which allows explore in a compact and effective (in the sense of machine computability) way the Hausdorff group of Lie exponentials (i.e. the shuffle characters) and special functions emerging from iterated integrals.

In particular, we have an analogue of Wei-Norman's theorem for these groups allowing to understand some multiplicative renormalisations (as those of Drinfeld). Parts of this work are strongly connected with Dyson series and take place within the project:

Evolution Equations in Combinatorics and Physics.

This talk also prepares data structures and spaces for Hoang Ngoc Minh's talk about associators.

An historic example : Lappo-Danilevskij's setting

J. A. Lappo-Danilevskij (J. A. Lappo-Danilevsky), Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I, *Travaux Inst. Physico-Math. Stekloff*, 1934, Volume 6, 1–256

§ 2. Hyperlogarithmes. En abordant la résolution algorithmique du problème de Poincaré, nous introduisons le système des fonctions

$$L_b(a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}|x), \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m;\ \nu = 1, 2, 3 \ldots)$$

définies par les relations de récurrence:

$$L_b(a_{j_1}|x) = \int_b^x \frac{dx}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}};$$

(10)

$$L_b(a_{j_1} a_{j_2} \ldots a_{j_\nu}|x) = \int_b^x \frac{L_b(a_{j_1} \ldots a_{j_{\nu-1}}|x)}{x - a_{j_\nu}}\, dx,$$

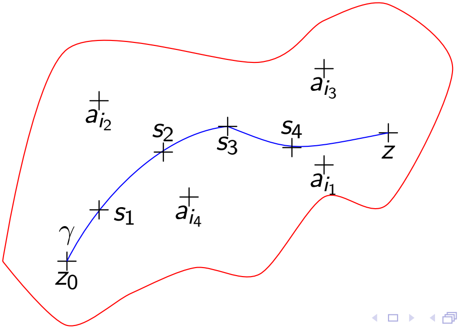
où b est un point fixe à distance finie, distinct des points $a_1, a_2, \ldots, a_m$. Ces fonctions seront nommées *hyperlogarithmes de la première espèce de*

Lappo-Danilevskij setting/2

Let $(a_i)_{1\le i\le n}$ be a family of complex numbers (all different) and $z_0 \notin \{a_i\}_{1\le i\le n}$, then

Definition [Lappo-Danilevskij, 1928]

$$L(a_{i_1},\ldots,a_{i_n}|z_0 \overset{\gamma}{\rightsquigarrow} z) = \int_{z_0}^{z}\int_{z_0}^{s_n}\ldots\left[\int_{z_0}^{s_1}\frac{ds}{s-a_{i_1}}\right]\ldots\frac{ds_n}{s_n-a_{i_n}}.$$



Remarks

1. The result depends only on the homotopy class of the path and then the result is a holomorphic function on $\widetilde{B}$ ($B = \mathbb{C} \setminus \{a_1, \cdots, a_n\}$)
2. From the fact that these functions are holomorphic, we can also study them in an open (simply connected) subset like the slit plane

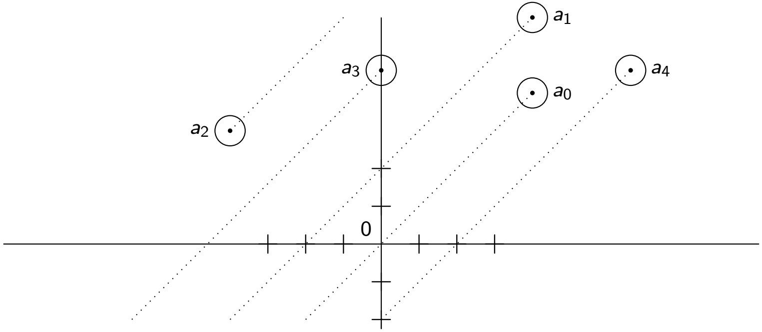


Figure: The slit plane (as cleft by half-rays).

Remarks/2

3. The set of functions $\alpha_{z_0}^{z}(\lambda) = L(a_{i_1}, \ldots, a_{i_n} | z_0 \overset{\gamma}{\rightsquigarrow} z)$ (or 1 if the list is void) has a lot of nice combinatorial properties
   - Noncommutative ED with left multiplier
   - Linear independence
   - Shuffle property
   - A Wei-Norman-like factorization in elementary exponentials
   - Possiblity of left or right multiplicative renormalization at a neighbourhood of the singularities
   - Extension to rational functions

In order to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about Non Commutative Formal Power Series[1], we will code the lists by words which will allow us to perform linear algebra and topology on the indexing.

[1] This was the initial intent of the series of conferences FPSAC.

Wei-Norman theorem

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Local coordinates on (infinite dimensional) Lie groups, factorization of Riemann zeta functions

3

3

Given a (finite dimensional) Lie group G (real $k=\mathbb{R}$ or complex $k=\mathbb{C}$) and its Lie algebra $\mathfrak{g}$, one can prove (a basis $B=(b_i)_{1\leq i\leq n}$ of $\mathfrak{g}$ being given) that there exists a neighbourhood W of 1_G (in G) and n *local coordinate* analytic functions

$$W\to k,\ (t_i)_{1\leq i\leq n}$$

such that, for all $g\in W$

$$(*)\quad g=\prod_{1\leq i\leq n}^{\rightarrow} e^{t_i(g)b_i}=e^{t_1(g)b_1}e^{t_2(g)b_2}\dots e^{t_n(g)b_n}$$

to see this, just remark that

$$(t_1,t_2,\cdots t_n)\to exp(t_1b_1)exp(t_2b_2)\cdots exp(t_nb_n)$$

is a *local* diffeomorphism from k^n to G in a neighbourhood of 0 and take the inverse.

This is the local **Wei-Norman's theorem**.

My questions are the following

Let us loosely call *infinite dimensional* a Lie group whose Lie algebra is not finite dimensional (this includes the example below and infinite dimensional Banach-Lie groups for instance).

Q1) Can you provide examples of infinite dimensional Lie groups where the exponential map

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Theorem (Wei-Norman theorem)

Let G be a k-Lie group (of finite dimension) ($k = \mathbb{R}$ or $k = \mathbb{C}$) and let $\mathfrak{g}$ be its k-Lie algebra. Let $B = \{b_i\}_{1 \leq i \leq n}$ be a (linear) basis of it. Then, there is a neighbourhood W of 1_G (within G) and n analytic functions (local coordinates)

$$W \to k,\ (t_i)_{1 \leq i \leq n}$$

such that, for all $g \in W$

$$g = \prod_{1 \leq i \leq n}^{\rightarrow} e^{t_i(g) b_i} = e^{t_1(g) b_1} e^{t_2(g) b_2} \ldots e^{t_n(g) b_n}.$$

Example

Example

We take $G = Gl_+(2,\mathbb{R})$ ($Gl_+(2,\mathbb{R})$, connected component of 1 within $Gl(2,\mathbb{R})$),

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad (1)$$

We will practically compute the Wei-Norman coefficients through an Iwasawa decomposition

$$M = \text{unitary x diagonal x triangular}$$

and compute $MTDU = I_2$ through the following elementary operations

1. (Orthogonalisation)
2. (Normalisation)
3. (Unitarisation)

$$
\begin{aligned}
M &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (C_1, C_2) = (C_1^{(1)}, C_2^{(1)}) e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \\
&= \underbrace{e^{arctan(\frac{a_{21}}{a_{11}}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}_{unitary} \underbrace{e^{log(||C_1^{(1)}||) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} e^{log(||C_2^{(1)}||) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}}_{diagonal\ \ (two\ \ exps)} \underbrace{e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}}_{triangular}
\end{aligned}
$$

We then get a Wei-Norman decomposition w.r.t. the following basis of $\mathfrak{gl}(2, \mathbb{R})$: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Use of this analogue for the group of characters

So, at the end of the day, if g is any shuffle character, we will get a factorization of the same type

$$g = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\langle g | S_l \rangle P_l} \ .$$

Let us now return to our iterated integrals.

Coding by words

Consider again the mapping

$$\alpha_{z_0}^{z}(\lambda) = L(a_{i_1}, \ldots, a_{i_n} | z_0 \overset{\gamma}{\rightsquigarrow} z) =: \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_n})$$

Lappo-Danilevskij recursion is from left to right, we will use here right to left indexing to match with [1, 2, 3, 4]. Data structures are there

1. Letters [1, 2]
2. Vector fields [3]
3. Matrices [4]

Words

We recall basic definitions and properties of the free monoid [5]:

- An alphabet is a set X (of variables or indeterminates, letters etc.)
- Words of length n (set X^n) are mappings $w : [1 \cdots n] \to X$. The letter at place j is $w[j]$, the empty word 1_{X^*} is the sole mapping $\emptyset \to X$ (i.e. of length 0). As such, we get, by composition, an action
  - of $\mathfrak{S}_n$ on the right (noted $w.\sigma$) and
  - of the transformation monoid X^X on the left
- Words concatenate by shifting and union of domains, this law is noted *conc*
- $(X^*, conc, 1_{X^*})$ is the free monoid of base X.
- Given a total order on X, (X^*) is totally ordered by the graded lexicographic ordering $\prec_{glex}$ (length first and then lexicographic from left to right). This ordering is compatible with the monoid structure.

Lyndon words and factorizations

- Let $c = [2 \cdots n, 1]$ be the large cycle
- a Lyndon word is a word which is strictly minimal in its conjugacy class (as a family) i.e. $(\forall k \in [1, n-1])(l \prec_{lex} l.\sigma^k)$
- Each word w factorizes uniquely as $w = l_1 l_2 \cdots l_n$ with $l_i \in \mathcal{L}yn(X)$ and $l_1 \succeq l_2 \cdots \succeq l_n$. $(\succeq = \succeq_{lex})$ We write

$$X^* = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} l^* \qquad (2)$$

- If $(P_l)_{l \in \mathcal{L}yn(X)}$ is any multihomogeneous basis of $Lie_R\langle X \rangle$ (R a $\mathbb{Q}$-algebra) then

$$\sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{S_l \otimes P_l}$$

where P_w is computed after eq. 2 and S_w is such that $\langle S_u | P_v \rangle = \delta_{u,v}$.

Noncommutative generating series

We now have a function $w \mapsto \alpha_{z_0}^{z}(w)$ which maps words to holomorphic functions on Ω. This is a noncommutative series of variables in X and coefficients in $\mathcal{H}(\Omega)$. It is convenient here to use the "sum notation".

$$S = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w$$

It is not difficult to see that S is the unique solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} \end{cases}$$

and that is it a shuffle character that is

$$\langle S | u \sqcup\!\sqcup v \rangle = \langle S | u \rangle \langle S | v \rangle \text{ and } \langle S | 1_{X^*} \rangle = 1$$

and, hence

$$S = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\langle S | S_l \rangle P_l} \, .$$

The series $S_{Pic}^{z_0}$

The series S can be computed by Picard's process

$$S_0 = 1_{X^*}\ ;\ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M(s).S_n(s)\,ds$$

and its limit is $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\,w$. One has,

Proposition

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \qquad \textbf{(DE)} \\ S(z_0) & = & 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases} \tag{3}$$

ii) The (complete) set of solutions of **(DE)** *is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.*

These (Noncommutative) Differential Equations with Multipliers (as eq. 3) admit a powerful calculus and set of properties .

Main facts about Non Commutative Diff. Eq.

Theorem

Let

$$(TSM) \qquad \mathbf{d}S = M_1 S + S M_2 \tag{4}$$

with $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X \rangle\rangle$

(i) *Solutions of* (TSM) *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* (TSM) *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

Theorem (cont'd)

(iv) *Let be the following one-sided equations*

$$(LM_1) \quad \mathbf{d}S = M_1 S \qquad (RM_2) \quad \mathbf{d}S = SM_2. \tag{5}$$

and let S_1 *(resp.* S_2*) be a solution of* (LM_1) *(resp.* (LM_2)*), then* S_1S_2 *is a solution of* (TSM)*. Conversely, every solution of* (TSM) *can be constructed so.*

(v) *Let* $S^{z_0}_{Pic,LM_1}$ *(resp.* $S^{z_0}_{Pic,RM_2}$*) the unique solution of* (LM_1) *(resp.* (RM_2)*) s.t.* $S(z_0) = 1_{\mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle}$ *then, the space of all solutions of* (TSM) *is*

$$S^{z_0}_{Pic,LM_1}.\mathbb{C}\langle\langle X\rangle\rangle.S^{z_0}_{Pic,RM_2}$$

(vi) *If* $M_i,\ i = 1,2$ *are primitive for* Δ_{III} *and if* S*, a solution of* (TSM)*, is group-like at one point (or asymptotically), it is group-like everywhere (over* Ω*).*

Linear (and algebraic) independence with combinatorics on words: Concrete form

Theorem (with Deneufchâtel and Solomon [6])

Let $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ be a solution of the (Left Multiplier) equation ($\mathcal{C} \subset \mathcal{H}(\Omega)$ a differential subfield)

$$\mathbf{d}(S) = MS \ ; \ \langle S|1_{X^*}\rangle = 1 \text{ with } M = \sum_{x\in X} u_x(z)\, x \in \mathcal{C}\langle\langle X\rangle\rangle$$

The following are equivalent :

i) *the family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients is independant (linearly) over $\mathcal{C}$.*

ii) *the family of coefficients $(\langle S|x\rangle)_{x\in X\cup\{1_{X^*}\}}$ is independant (linearly) over $\mathcal{C}$.*

iii) *the family $(u_x)_{x\in X}$ is such that, for $f \in \mathcal{C}$ et $\alpha_x \in \mathbb{C}$*

$$\mathrm{d}(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

Linear independence by combinatorics on words: Abstract theorem

Theorem ([6])

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0,\ ker(d) = k$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = M.S\ ;\ \langle S|1\rangle = 1 \tag{6}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle\ . \tag{7}$$

Then, the following conditions are equivalent :

Abstract theorem/2

Theorem (cont'd)

1. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
2. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
3. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{8}$$

4. *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{9}$$

Sketch of the proof (or goto slide 20)

(i)$\Longrightarrow$(ii) Obvious.
(ii)$\Longrightarrow$(iii)

- suppose free (over $\mathcal{C}$) the family $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$.
- consider a relation $d(f)=\sum_{x\in X}\alpha_x u_x$
- form a formal pattern of this relation $P=-f1_{X^*}+\sum_{x\in X}x$
- differentiate $\langle S|P\rangle$ and obtain $\langle S|P\rangle=\lambda\in k$
- form $Q=P-\lambda.1_{X^*}=-(f+\lambda)1_{X^*}+\sum_{x\in X}\alpha_x x$ and from $\langle S|Q\rangle=0$, get all $\alpha_x=0$.

(iii)$\Longleftrightarrow$(iv)
Obvious, (iv) being a geometric reformulation of (iii).
(iii)$\Longrightarrow$(i)
Let $\mathcal{K}$ be the kernel of $P\mapsto\langle S|P\rangle$ (a form $\mathcal{C}\langle X\rangle\to\mathcal{A}$) i.e.

$$\mathcal{K}=\{P\in\mathcal{C}\langle X\rangle|\langle S|P\rangle=0\}\ . \tag{10}$$

If $\mathcal{K}=\{0\}$, we are done. Otherwise, let us adopt the following strategy.

First, we order X by some well-ordering $<$ and X^* by the graded lexicographic ordering $\prec$ defined by

$$u \prec v \iff |u| < |v| \text{ or } (u = pxs_1\ ,\ v = pys_2 \text{ and } x < y). \tag{11}$$

It is easy to check that $\prec$ is also a well-ordering relation. For each nonzero polynomial P, we denote by $lead(P)$ its leading monomial; i.e. the greatest element of its support $\mathrm{supp}(P)$ (for $\prec$).
Now, as $\mathcal{R} = \mathcal{K} - \{0\}$ is not empty, let w_0 be the minimal element of $lead(\mathcal{R})$ and choose a $P \in \mathcal{R}$ such that $lead(P) = w_0$. We write

$$P = fw_0 + \sum_{u \prec w_0} \langle P|u\rangle u\ ;\ f \in \mathcal{C} - \{0\}\ . \tag{12}$$

The polynomial $Q = \frac{1}{f}P$ is also in $\mathcal{R}$ with the same leading monomial, but the leading coefficient is now 1; and so Q is given by

$$Q = w_0 + \sum_{u \prec w_0} \langle Q|u\rangle u\ . \tag{13}$$

Differentiating $\langle S|Q\rangle = 0$, one gets

$$\begin{aligned} 0 &= \langle \mathbf{d}(S)|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle MS|Q\rangle + \langle S|\mathbf{d}(Q)\rangle \\ &= \langle S|M^\dagger Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle S|M^\dagger Q + \mathbf{d}(Q)\rangle \end{aligned} \tag{14}$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x\in X} u_x(x^\dagger Q) + \sum_{u\prec w_0} d(\langle Q|u\rangle)u \in \mathcal{C}\langle X\rangle \ . \tag{15}$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0, hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q|u\rangle) = -\sum_{x\in X} u_x\langle Q|xu\rangle \ ; \ \text{for } x \in X \ , \ v \in X^*. \tag{16}$$

From this last relation, we deduce that $\langle Q|w\rangle \in k$ for every w of length $deg(Q)$ and, from $(\langle S|1\rangle = 1,\ \langle S|Q\rangle = 0)$, one must have $deg(Q) > 0$. Then, we write $w_0 = yv$ and compute the coefficient at v

$$d(\langle Q|v\rangle) = -\sum_{x\in X} u_x\langle Q|xv\rangle = \sum_{x\in X} \alpha_x u_x \tag{17}$$

with coefficients $\alpha_x = -\langle Q|xv\rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition (8) implies that all coefficients $\langle Q|xu\rangle$ are zero ; in particular, as $\langle Q|yu\rangle = \langle Q|w_0\rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$. $\square$

Example (See [9] for $\mathcal{C} = \mathbb{C}$)

$$\mathbf{d}(S) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)S\ ;\ S(z_0) = 1\ ;\ z_0 \in \Omega\ .$$

with $\mathcal{C} = \mathbb{C}(z)$ (germs) and $\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$.

Solutions as $\shuffle$-characters with values in $\mathcal{H}(\Omega)$

We have seen that (some) solutions of systems like that of Hyperlogarithms possess the shuffle property i.e. defining the shuffle product by the recursion

$$\begin{aligned} u \shuffle 1_{Y^*} &= 1_{Y^*} \shuffle u = u \text{ and} \\ au \shuffle bv &= a(u \shuffle bv) + b(au \shuffle v) \end{aligned}$$

one has

$$\langle S_{Pic}^{z_0} | u \shuffle v \rangle = \langle S_{Pic}^{z_0} | u \rangle \langle S_{Pic}^{z_0} | v \rangle \ ; \ \langle S_{Pic}^{z_0} | 1_{X^*} \rangle = 1 \tag{18}$$

(product in $\mathcal{H}(\Omega)$).

Now it is not difficult to check that the characters of type (18) form a group (these are characters on a Hopf algebra). It is interesting to have at our disposal a system of local coordinates in order to perform estimates in neighbourhood of the singularities.

Schützenberger's (MRS) factorisation

This MRS[2] factorisation is, in fact, a resolution of the identity. It reads as follows

Theorem (Schützenberger, 1958, Reutenauer, 1988)

Let $\mathcal{D}_X := \sum_{w\in X^*} w\otimes w$. *Then* $\mathcal{D}_X = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{S_l\otimes P_l}$.

where the product laws is the shuffle on the left and concatenation on the right, $(P_l)_{l\in\mathcal{L}yn(X)}$ is an homogeneous basis of $\mathcal{L}ie\langle X\rangle$ and $(S_l)_{l\in\mathcal{L}yn(X)}$, the "Lyndon part" of the dual basis of $(P_w)_{w\in X^*}$ which, given that is formed by

$$P_{l_1}^{\alpha_1}\ldots P_{l_n}^{\alpha_n} \text{ where } w = l_1^{\alpha_1}\ldots l_n^{\alpha_n} \text{ with } l_1 > \ldots > l_n \text{ (lexorder)}$$

[2] after Mélançon, Reutenauer, Schützenberger

Applying MRS to a shuffle character

Now, remarking that this factorization lives within the subalgebra

$$Iso(X) = \{T \in R\langle\langle X^* \otimes X^* \rangle\rangle | (u \otimes v \in supp(T) \Longrightarrow |u| = |v|)\}$$

if Z is any shuffle character, one has

$$Z = (Z \otimes Id)(\sum_{w \in X^*} w \otimes w) = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\langle Z | S_l \rangle P_l}$$

We would like to get such a factorisation at our disposal for other types of (deformed) shuffle products, this will be done in the second part of the talk. Let us first, with this factorization (MRS) at hand, construct explicitely Drinfeld's solution G_0.

Extensions of MRS to other shuffles

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$	$\varphi \equiv 0$	I
Stuffle [19]	$x_i u \stuffle x_j v = x_i(u \stuffle x_j v) + x_j(x_i u \stuffle v)$ $+ x_{i+j}(u \stuffle v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [7]	$x_i u \minstuffle x_j v = x_i(u \minstuffle x_j v) + x_j(x_i u \minstuffle v)$ $- x_{i+j}(u \minstuffle v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [14]	$x_i u \muffle x_j v = x_i(u \muffle x_j v) + x_j(x_i u \muffle v)$ $+ x_{i\times j}(u \muffle v)$	$\varphi(x_i, x_j) = x_{i\times j}$	I
q-shuffle [3]	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v)$ $+ q x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q-shuffle$_2$	$x_i u \stuffle_q x_j v = x_i(u \stuffle_q x_j v) + x_j(x_i u \stuffle_q v)$ $+ q^{i.j} x_{i+j}(u \stuffle_q v)$	$\varphi(x_i, x_j) = q^{i.j} x_{i+j}$	II
$\mathtt{LDIAG}(1, q_s)$ [10] (non-crossed, non-shifted)	$au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v)$ $+ q_s^{\lvert a\rvert\lvert b\rvert} a.b(u \shuffle v)$	$\varphi(a, b) = q_s^{\lvert a\rvert\lvert b\rvert}(a.b)$	II
q-Infiltration [12]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v)$ $+ q\delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q\delta_{a,b} a$	III
AC-stuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	IV
Semigroup-stuffle	$x_t u \shuffle_\perp x_s v = x_t(u \shuffle_\perp x_s v) + x_s(x_t u \shuffle_\perp v)$ $+ x_{t\perp s}(u \shuffle_\perp v)$	$\varphi(x_t, x_s) = x_{t\perp s}$	I
φ-shuffle	$au \shuffle_\varphi bv = a(u \shuffle_\varphi bv) + b(au \shuffle_\varphi v)$ $+ \varphi(a, b)(u \shuffle_\varphi v)$	$\varphi(a, b)$ law of AAU	V

Of course, the q-shuffle is equal to the (classical) shuffle when $q = 0$. As for the q-

About Drinfeld's solutions G_0, G_1

We give below the computational construction of a solution with an asymptotic condition.
In his paper (2. above), V. Drinfel'd states that there is a unique solution (called G_0) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

and a unique solution (called G_1) of

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S \\ \lim_{\substack{z\to 1\\ z\in\Omega}} e^{x_1 log(1-z)} S(z) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases}$$

Let us give here, as an example, a construction of G_0 (G_1 can be derived or checked by symmetry see also Minh's talk).

Explicit construction of Drinfeld's G_0

Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_0^s(u)\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0\ . \end{cases} \tag{19}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$

satisfies the one sided evolution equation (LM)

$$\mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S$$

hence $T = (\sum_{w \in X^*} \alpha_0^z(w)\, w)e^{-x_0 log(z)}$ satisfies the two sided evolution equation (TSM)

$$\mathbf{d}(T) = (\frac{x_0}{z} + \frac{x_1}{1-z}).T + T.(-\frac{x_0}{z})$$

Now, using Radford's theorem, one proves that S is group-like, factorizes through (MRS) and that $\lim_{z \to 0} T(z) = 1$.
This asymptotic condition on T implies that $S = G_0$.

A remark

One can modify construction (19) using $t \in \Omega$ instead of 1 as follows

Remark

$$\alpha_t^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \alpha_t^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_t^z \alpha_t^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \alpha_t^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ . \end{cases}$$

One still has that $G(t) := \sum_{w \in X^*} \alpha_t^z(w)\, w$ is group-like (similar proof) and $\lim_{t \to 1} G(t) = G_0$ as

$$G(t) = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{\langle G(t) | S_l \rangle P_l} = \Big(\prod_{l \in \mathcal{L}yn(X) \setminus \{x_0\}}^{\searrow} e^{\langle G(t) | S_l \rangle P_l} \Big) e^{x_0 (log(z) - log(t))}$$

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation.
- MRS factorisation provides an analogue of the (local) theorem of Wei-Norman and allows to remove singularities with simple counterterms.
- MRS factorisation can be performed in many other cases (like stuffle for harmonic functions)
- Use of combinatorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld's G_0 has a domain which includes only some rational series (cf Minh's talk).

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Thank you for your attention.