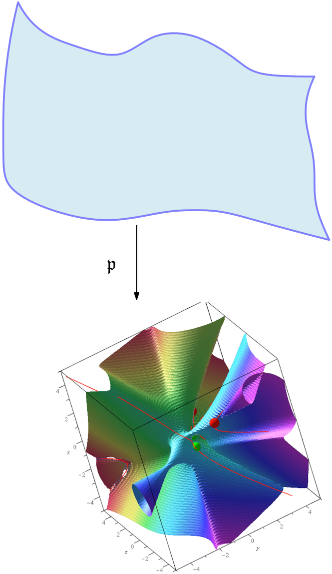
PROJECTION OF ANALYTIC SURFACES

S. Diatta, G. Moroz and M. Pouget

JNCF January 22, 2017

Description

- $F, G : \mathbb{R}^4 \to \mathbb{R}$ two real analytic functions
- $\mathcal{M} = \{q \in \mathbb{R}^4 | F(q) = G(q) = 0\}$ be a smooth surface
- $\mathfrak{p} : \mathbb{R}^4 \to \mathbb{R}^3$, $(x, y, z, t) \mapsto (x, y, z)$
- $\Omega = \mathfrak{p}(\mathcal{M})$ can be a singular surface
- $\Omega = \Omega_{reg} \cup \Omega_{sing}$



Contribution: Graph of singularities of $\Omega (\Omega_{sing})$
Future work: Compute a triangulation isotopic to Ω.

Outline

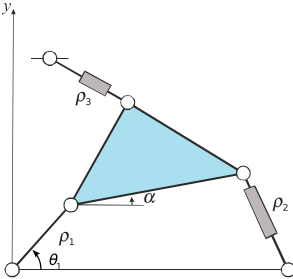
1. Motivation and Approach
2. Types of singularity
   - Generic singularities
   - Characterization with regular systems
3. Sub-algorithms
4. Topology of singularities

Motivation in Robotic

- $2RPR - RR$: parallel mechanism
- ρ_1 fixed
- Articular variables: ρ_2, ρ_3
- Pose variables: θ_1, α

E_W is a smooth variety of dimension 2 contained in a 4-dimensional space.

The projection of E_W along one direction provides a visualization



Outline

1. Motivation and Approach
2. Types of singularity
   - Generic singularities
   - Characterization with regular systems
3. Sub-algorithms
4. Topology of singularities

About classification of singularities

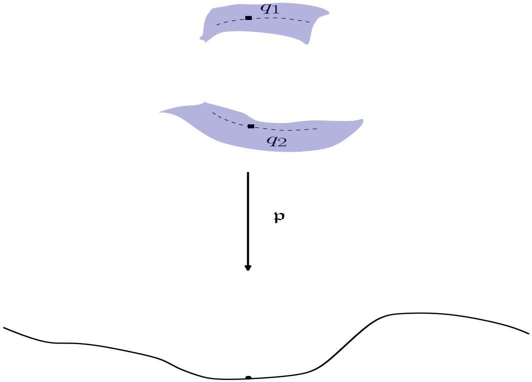
In mathematics the are several results about the germs of maps: **[Whit44]**, **[Whit55]**, **[Math69]**, **[Arno81]**, **[Mond83]**, **[Gor84]**, **[Rieg86]**, **[MaTa96]** ...

but, there is less work about multi-germs maps:

- **[Gor96]** Victor V. Goryunov : *Local invariants of mappings of surfaces into three-space*
- **[HK01]** C. A. Hobbs and N. P. Kirk: *On the classification and bifurcation of multigerms of maps from surfaces to 3-space*.

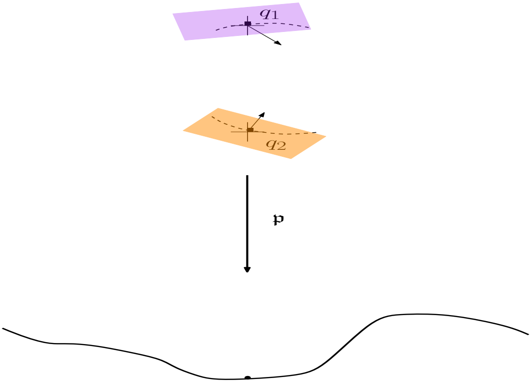
double-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



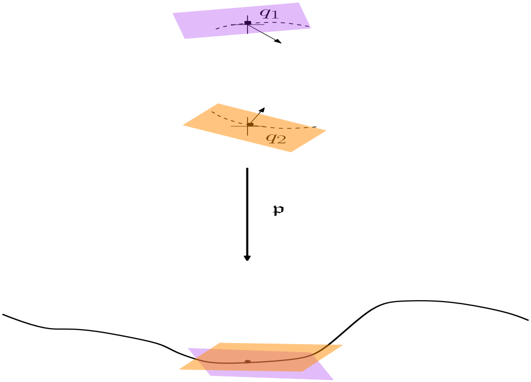
double-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



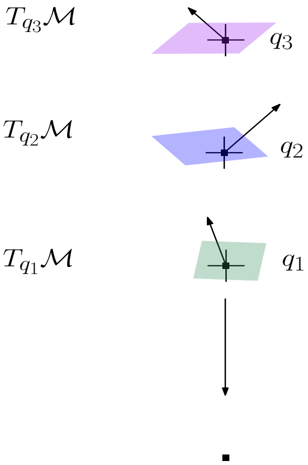
double-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



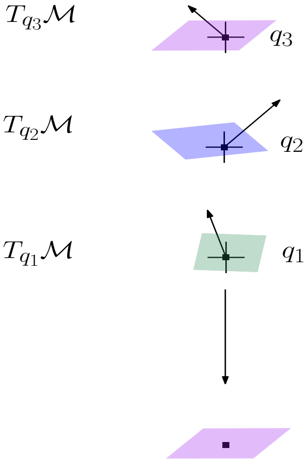
triple-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



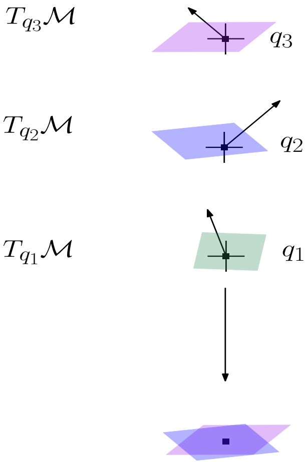
triple-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



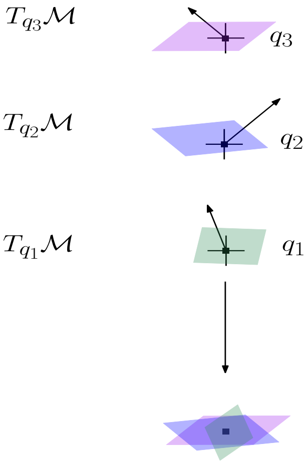
triple-point

[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



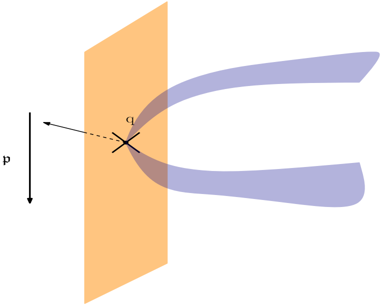
triple-point

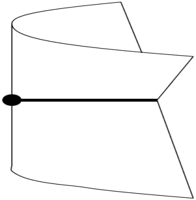
[Gor96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



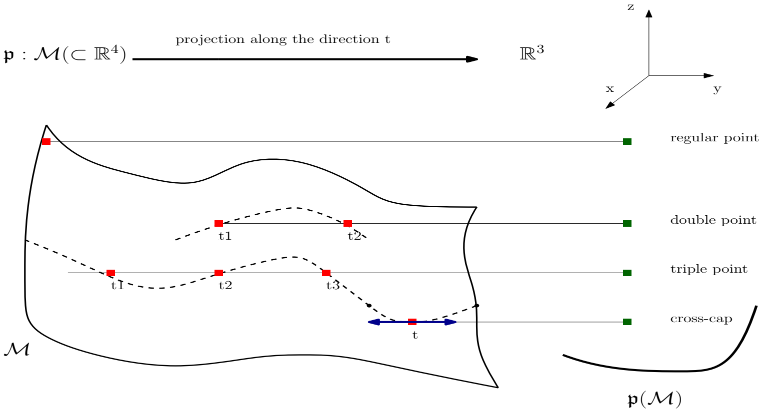
cross-cap

[Gor,96] Victor V. Goryunov: *Local invariants of mappings of surfaces into three-space*.



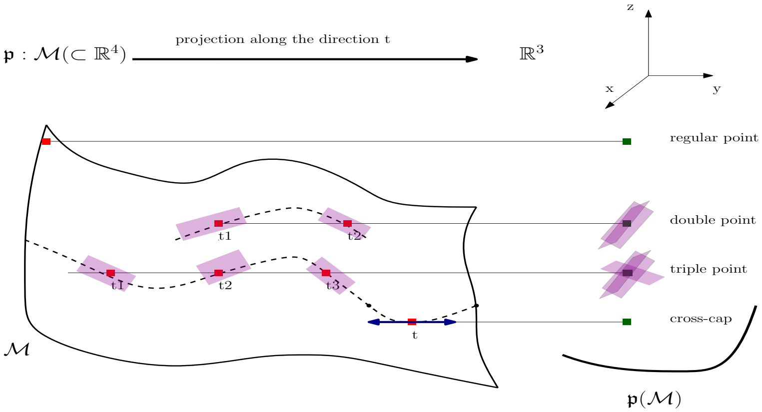


Assumptions



Different types of singularities of $\mathfrak{p}(\mathcal{M})$ with their preimages

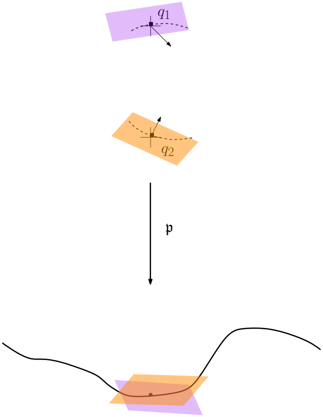
Assumptions



Different types of singularities of $\mathfrak{p}(\mathcal{M})$ with their preimages

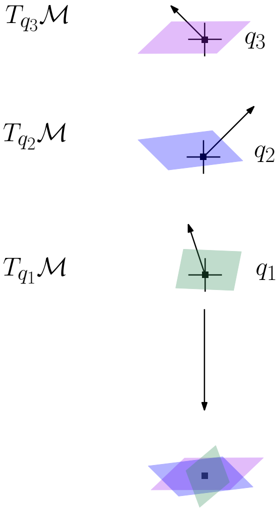
curve of double-points

$$(S_{double}) \begin{cases} F(x,y,z,t_1) = 0 \\ G(x,y,z,t_1) = 0 \\ F(x,y,z,t_2) = 0 \\ G(x,y,z,t_2) = 0 \\ \text{with } t_1 \neq t_2 \end{cases}$$



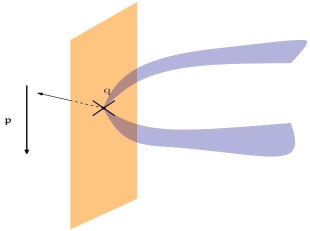
triple-point

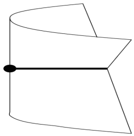
$$
(S_{triple}) \begin{cases} F(x,y,z,t_1) = 0 \\ G(x,y,z,t_1) = 0 \\ F(x,y,z,t_2) = 0 \\ G(x,y,z,t_2) = 0 \\ F(x,y,z,t_3) = 0 \\ G(x,y,z,t_3) = 0 \\ t_i \neq t_j \text{ when } i \neq j \end{cases}
$$



cross-cap

$$(S_{cross}) \begin{cases} F(x,y,z,t) = 0 \\ G(x,y,z,t) = 0 \\ \partial_t F(x,y,z,t) = 0 \\ \partial_t G(x,y,z,t) = 0 \end{cases}$$





Results

Assumptions

(i) $\mathcal{M}$ is smooth;

(ii) $P \in \Omega$ has at most three pre-images;

(iii) If $P \in \Omega$ has three pre-images, then all of them are regular;

(iv) The tangent plans have complete intersection.

Theorem (Dimensions)

The set of solutions of (S_{double}) *is* 1*-dimensional and the set of solutions of* (S_{triple}) *and* (S_{cross}) *are each of them* 0*-dimensional.*

Theorem (Regularity)

The systems (S_{double}), (S_{triple}) *and* (S_{cross}) *are regular if the above assumptions are satisfied.*

Remark

(S_{double}) *becomes non regular when* $t_1 = t_2$.

Results

Assumptions

(i) $\mathcal{M}$ is smooth;

(ii) $P \in \Omega$ has at most three pre-images;

(iii) If $P \in \Omega$ has three pre-images, then all of them are regular;

(iv) The tangent plans have complete intersection.

Theorem (Dimensions)

The set of solutions of (S_{double}) is 1-dimensional and the set of solutions of (S_{triple}) and (S_{cross}) are each of them 0-dimensional.

Theorem (Regularity)

The systems (S_{double}), (S_{triple}) and (S_{cross}) are regular if the above assumptions are satisfied.

Remark

(S_{double}) becomes non regular when $t_1 = t_2$.

Results

Assumptions

(i) $\mathcal{M}$ is smooth;

(ii) $P \in \Omega$ has at most three pre-images;

(iii) If $P \in \Omega$ has three pre-images, then all of them are regular;

(iv) The tangent plans have complete intersection.

Theorem (Dimensions)

The set of solutions of (S_{double}) is 1-dimensional and the set of solutions of (S_{triple}) and (S_{cross}) are each of them 0-dimensional.

Theorem (Regularity)

The systems (S_{double}), (S_{triple}) and (S_{cross}) are regular if the above assumptions are satisfied.

Remark

(S_{double}) becomes non regular when $t_1 = t_2$.

Results

Assumptions

(i) $\mathcal{M}$ is smooth;

(ii) $P \in \Omega$ has at most three pre-images;

(iii) If $P \in \Omega$ has three pre-images, then all of them are regular;

(iv) The tangent plans have complete intersection.

Theorem (Dimensions)

The set of solutions of (S_{double}) *is* 1*-dimensional and the set of solutions of* (S_{triple}) *and* (S_{cross}) *are each of them* 0*-dimensional.*

Theorem (Regularity)

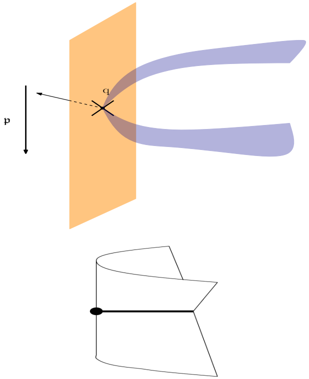
The systems (S_{double}), (S_{triple}) *and* (S_{cross}) *are regular if the above assumptions are satisfied.*

Remark

(S_{double}) *becomes non regular when* $t_1 = t_2$.

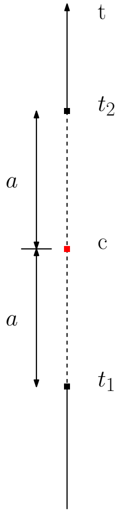
curve of double-points

$$(S_{double}) \begin{cases} F(x,y,z,t_1) = 0 \\ G(x,y,z,t_1) = 0 \\ F(x,y,z,t_2) = 0 \\ G(x,y,z,t_2) = 0 \\ \text{with } t_1 \neq t_2 \end{cases}$$



Ball system

We consider the following change of variables:



- For any two points $q_1 = (x, y, z, t_1)$ and $q_2 = (x, y, z, t_2)$ on $\mathcal{M}$,
- $c = \frac{t_1+t_2}{2}$ and $r = a^2$

$$(S_{ball})_{r>0} \begin{cases} \frac{1}{2}(F(x,y,z,c+\sqrt{r}) + F(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2}(G(x,y,z,c+\sqrt{r}) + G(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2\sqrt{r}}(F(x,y,z,c+\sqrt{r}) - F(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2\sqrt{r}}(G(x,y,z,c+\sqrt{r}) - G(x,y,z,c-\sqrt{r})) = 0 \end{cases}$$

$$(S_{ball})_{r=0} \begin{cases} F(x,y,z,c) = 0 \\ G(x,y,z,c) = 0 \\ \partial_t F(x,y,z,c) = 0 \\ \partial_t G(x,y,z,c) = 0 \end{cases}$$

$\pi : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

Lemma

1. $\pi(Sol_{ball}) = \mathfrak{p}(Sol_{double}) \cup \mathfrak{p}(Sol_{cross})$
2. (S_{ball}) *is regular if the assumptions are satisfied*

$$
(S_{ball})_{r>0} \begin{cases} \frac{1}{2}(F(x,y,z,c+\sqrt{r}) + F(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2}(G(x,y,z,c+\sqrt{r}) + G(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2\sqrt{r}}(F(x,y,z,c+\sqrt{r}) - F(x,y,z,c-\sqrt{r})) = 0 \\ \frac{1}{2\sqrt{r}}(G(x,y,z,c+\sqrt{r}) - G(x,y,z,c-\sqrt{r})) = 0 \end{cases}
$$

$$
(S_{ball})_{r=0} \begin{cases} F(x,y,z,c) = 0 \\ G(x,y,z,c) = 0 \\ \partial_t F(x,y,z,c) = 0 \\ \partial_t G(x,y,z,c) = 0 \end{cases}
$$

$\pi : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$

Lemma

1. $\pi(Sol_{ball}) = \mathfrak{p}(Sol_{double}) \cup \mathfrak{p}(Sol_{cross})$
2. (S_{ball}) *is regular if the assumptions are satisfied*

Outline

1. Motivation and Approach
2. Types of singularity
   - Generic singularities
   - Characterization with regular systems
3. Sub-algorithms
4. Topology of singularities

Interval Newton Method

Let

- $\mathcal{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 function;
- X be a product of intervals in $\mathbb{R}^n$ and $q_0 \in X$;
- $J_{\mathcal{F}(X)}$ be the Jacobian matrix of $\mathcal{F}$ in X;
- $[J_{\mathcal{F}(X)}]$ the interval enclosure of $J_{\mathcal{F}(X)}$.

Interval Newton operator is defined by:

$$N(q_0, X) = q_0 - [J_{\mathcal{F}(X)}]^{-1}\mathcal{F}(q_0).$$

Existence and uniqueness of solution

1. If $N(q_0, X) \subset X$, then $\exists! q' \in X$ such that $F(q') = 0$.
2. If $q" \in X$ and $\mathcal{F}(q") = 0$, then $q" \in N(q_0, X)$.
3. If $N(q_0, X) \cap X = \emptyset$, then $\mathcal{F}(q) \neq 0$ for all $q \in X$.

Interval Newton Method

Let

- $\mathcal{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1 function;
- X be a product of intervals in $\mathbb{R}^n$ and $q_0 \in X$;
- $J_{\mathcal{F}(X)}$ be the Jacobian matrix of $\mathcal{F}$ in X;
- $[J_{\mathcal{F}(X)}]$ the interval enclosure of $J_{\mathcal{F}(X)}$.

Interval Newton operator is defined by:

$$N(q_0, X) = q_0 - [J_{\mathcal{F}(X)}]^{-1}\mathcal{F}(q_0).$$

Existence and uniqueness of solution

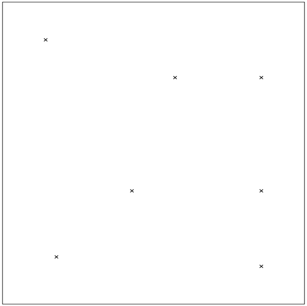
1. If $N(q_0, X) \subset X$, then $\exists! q' \in X$ such that $F(q') = 0$.
2. If $q" \in X$ and $\mathcal{F}(q") = 0$, then $q" \in N(q_0, X)$.
3. If $N(q_0, X) \cap X = \emptyset$, then $\mathcal{F}(q) \neq 0$ for all $q \in X$.

Sub-algorithms

[Neu90] A. Neumaier: *Interval methods for systems of equations.*

`IsolatBoxes` **algorithm**: Isolating boxes for a regular 0-dimensional system

- Input: (S, X_0) with $X_0 \subset \mathbb{R}^n$
- Output: A set X^{sol} of isolating boxes pairwise disjoint



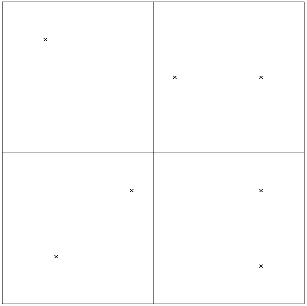
JNCF January 22, 2017

Sub-algorithms

[Neu90] A. Neumaier: *Interval methods for systems of equations.*

`IsolatBoxes` **algorithm**: Isolating boxes for a regular 0-dimensional system

- Input: (S, X_0) with $X_0 \subset \mathbb{R}^n$
- Output: A set X^{sol} of isolating boxes pairwise disjoint

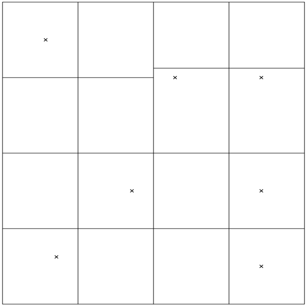


Sub-algorithms

[Neu90] A. Neumaier: *Interval methods for systems of equations.*

`IsolatBoxes` **algorithm**: Isolating boxes for a regular 0-dimensional system

- Input: (S, X_0) with $X_0 \subset \mathbb{R}^n$
- Output: A set X^{sol} of isolating boxes pairwise disjoint

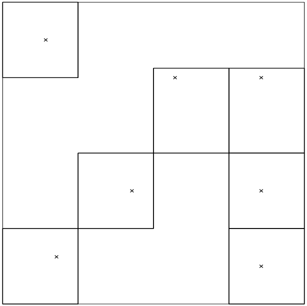


Sub-algorithms

[Neu90] A. Neumaier: *Interval methods for systems of equations.*

`IsolatBoxes` **algorithm**: Isolating boxes for a regular 0-dimensional system

- Input: (S, X_0) with $X_0 \subset \mathbb{R}^n$
- Output: A set X^{sol} of isolating boxes pairwise disjoint

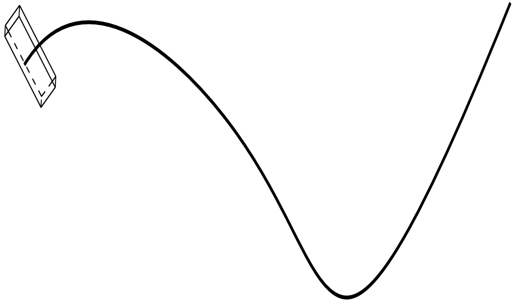


Sub-algorithms

C Beltràn and A Leykin [2012], **B Martin A Goldsztejn L Granvilliers and C Jermann [2013]**, **J V D Hoeven [2015]**

`Curve Tracking Reliable` **algorithm**: Compute a sequence of *n*-dimensional parallelotopes enclosing of a given connected component.

- Input: Initial point
- Output: A set of adjacent parallelotopes enclosing each connected component

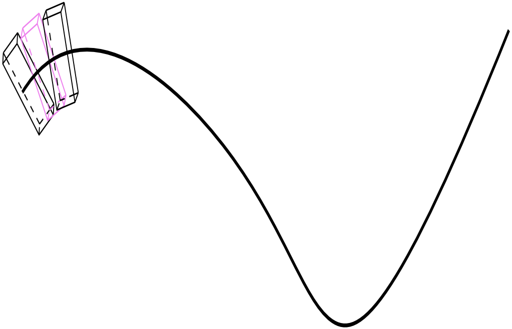


Sub-algorithms

C Beltràn and A Leykin [2012], **B Martin A Goldsztejn L Granvilliers and C Jermann [2013]**, **J V D Hoeven [2015]**

`Curve Tracking Reliable` **algorithm**: Compute a sequence of *n*-dimensional parallelotopes enclosing of a given connected component.

- Input: Initial point
- Output: A set of adjacent parallelotopes enclosing each connected component

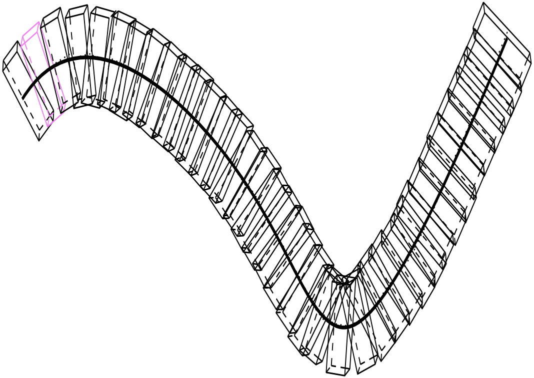


Sub-algorithms

C Beltràn and A Leykin [2012], B Martin A Goldsztejn L Granvilliers and C Jermann [2013], J V D Hoeven [2015]

`Curve Tracking Reliable` **algorithm**: Compute a sequence of n-dimensional parallelotopes enclosing of a given connected component.

- Input: Initial point
- Output: A set of adjacent parallelotopes enclosing each connected component



Outline

1 Motivation and Approach

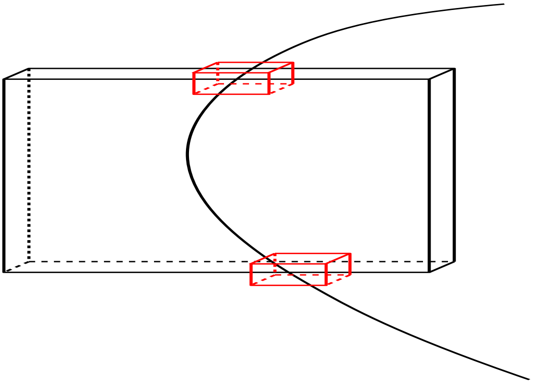
2 Types of singularity
- Generic singularities
- Characterization with regular systems

3 Sub-algorithms

4 Topology of singularities

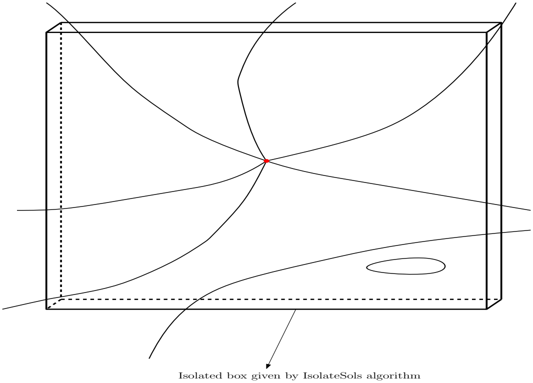
Enclose special points of Ω_{sing}

- x-critical points: `IsolatBoxes`$((S_{ball})', X_0)$ provided a set of boxes in $\mathbb{R}^5$,
- `IsolatBoxes`$((S_{triple}), X_0)$ provided a set of boxes in $\mathbb{R}^6$,
- `IsolatBoxes`$((S_{cross}), X_0)$ provided a set of boxes in $\mathbb{R}^4$ and
- boundary points.



Enclose special points of Ω_{sing}

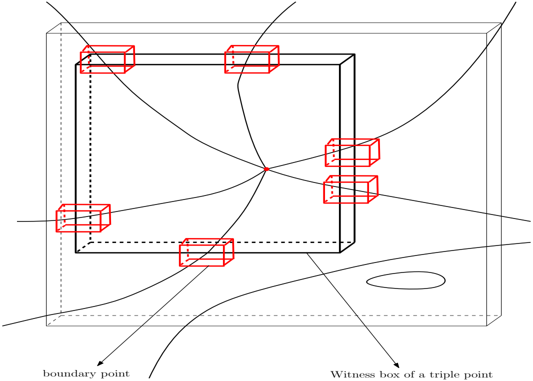
- x-critical points: `IsolatBoxes`$((S_{ball})', X_0)$ provided a set of boxes in $\mathbb{R}^5$,
- `IsolatBoxes`$((S_{triple}), X_0)$ provided a set of boxes in $\mathbb{R}^6$,
- `IsolatBoxes`$((S_{cross}), X_0)$ provided a set of boxes in $\mathbb{R}^4$ and
- boundary points.



Compute witness boxes

B is called *witness* box if it satisfied the following conditions:

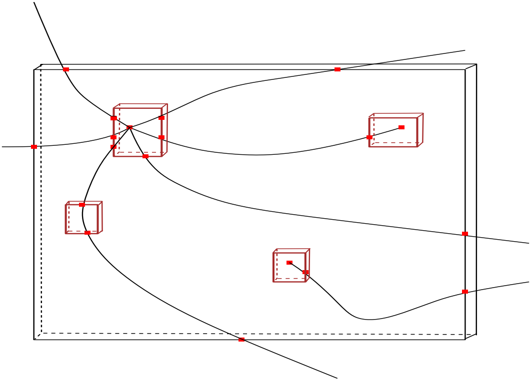
1. B is the projection in $\mathbb{R}^3$ of a isolating box
2. B doesn't contain any x-critical point
3. $B \cap \mathfrak{p}(X_i) = \emptyset$, with X_i is an isolating box



Path tracking

Compute a set of parallelotopes that enclose each connected component

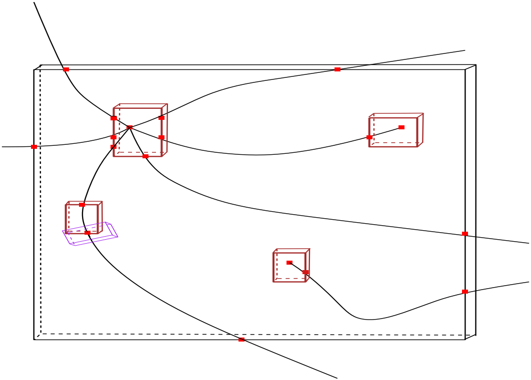
- Find at least one point on each connected component: x-critical point, boundary point, cross-cap
- start the path tracking at these point



Path tracking

Compute a set of parallelotopes that enclose each connected component

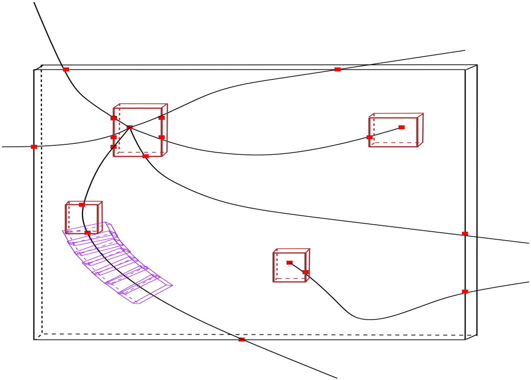
- Find at least one point on each connected component: x-critical point, boundary point, cross-cap
- start the path tracking at these point



Path tracking

Compute a set of parallelotopes that enclose each connected component

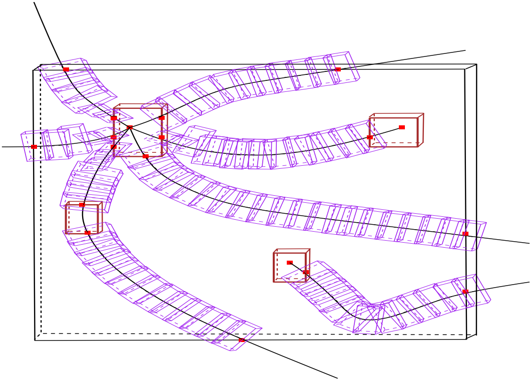
- Find at least one point on each connected component: x-critical point, boundary point, cross-cap
- start the path tracking at these point



Path tracking

Compute a set of parallelotopes that enclose each connected component

- Find at least one point on each connected component: x-critical point, boundary point, cross-cap
- start the path tracking at these point



THANK YOU!