Regularity and Gröbner bases of the Rees algebra of edge ideals of bipartite graphs

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Definition

A **bipartite graph** $G = (X, Y, E)$ consists of two disjoint sets of vertices $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and a set of edges

$$E \subset \{(x, y) \mid x \in X, y \in Y\}.$$

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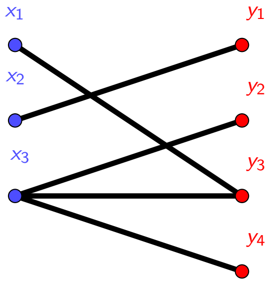
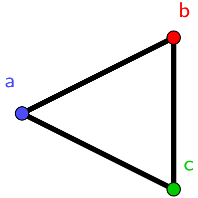
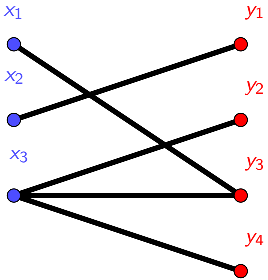




Definition

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. The **edge ideal** $I = I(G)$, associated to G, is defined by

$$I = \left(x_i y_j \mid (x_i, y_j) \in E\right).$$



$$I = \left(x_1 y_3, x_2 y_1, x_3 y_2, x_3 y_3, x_3 y_4\right) \subset R$$

Definition

Let $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^i t^i \subset R[t]$ be the **Rees algebra** of the edge ideal I. Let $f_1, \ldots, f_q$ be the square free monomials of degree two generating I. Let $S = R[T_1, \ldots, T_q]$, and define the following map

$$S = \mathbb{K}[x_1, \ldots, x_n, y_1 \ldots, y_m, T_1, \ldots, T_q] \xrightarrow{\psi} \mathcal{R}(I) \subset R[t],$$

$$\psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_i) = f_i t.$$

Then the presentation of $\mathcal{R}(I)$ is given by $S/\mathcal{K}$ where $\mathcal{K} = \mathrm{Ker}(\psi)$.

Problem

In terms of the combinatorics of the **bipartite graph** G, we want to:

- Describe the universal Gröbner basis of $\mathcal{K}$.
- Compute the Castelnuovo-Mumford regularity of $\mathcal{R}(I)$.
- Study the regularity of the powers of the ideal I.

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Matrix associated to the presentation of $\mathcal{R}(I)$

Given the presentation of the Rees algebra $\psi : S \to \mathcal{R}(I)$

$$\psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_i) = f_i t.$$

Let $A = (a_{i,j}) \in \mathbb{Z}^{n+m,q}$ be the incidence matrix of G, i.e. each column corresponds to an edge f_i. Then we construct the following matrix

$$M = \begin{array}{c} \\ \left(\begin{array}{ccccccccc} f_1 t & \cdots & f_q t & x_1 & \cdots & x_n & y_1 & \cdots & y_m \\ a_{1,1} & \cdots & a_{1,q} & \mathbf{e_1} & \cdots & \mathbf{e_n} & \mathbf{e_{n+1}} & \cdots & \mathbf{e_{n+m}} \\ \vdots & \ddots & \vdots & & & & & & \\ a_{n+m,1} & \cdots & a_{n+m,q} & & & & & & \\ 1 & \cdots & 1 & & & & & & \end{array} \right) \end{array}$$

$\mathcal{K}$ is a toric ideal (Sturmfels 1996)

$$\mathcal{K} = \left(\mathbf{Txy}^{\alpha^+} - \mathbf{Txy}^{\alpha^-} \mid \alpha \in \mathrm{Ker}_{\mathbb{Z}}(M) \right)$$

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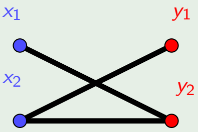

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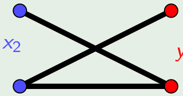

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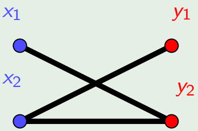

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Universal Gröbner basis of $\mathcal{K}$

$$\mathcal{U} = \bigcup_{<\text{ runs over all possible term orders}} \mathcal{G}_<(\mathcal{K})$$

($\mathcal{G}_<(\mathcal{K})$ denotes reduced Gröbner basis with respect to $<$)

Circuit

$\alpha \in \mathrm{Ker}_{\mathbb{Z}}(M)$ is called a circuit if it has minimal support $\mathrm{supp}(\alpha)$ with respect to inclusion and its coordinates are relatively prime.

In general we have that the set of circuits is contained in $\mathcal{U}$.

Lemma

If G is a bipartite graph then $\mathcal{U} = \left\{ \mathbf{Txy}^{\alpha^+} - \mathbf{Txy}^{\alpha^-} \mid \alpha \text{ is a circuit of } M \right\}$.

Proof.

From Gitler, Valencia, and Villarreal 2005, then M is totally unimodular. Hence, by Sturmfels 1996 we get the equality. □

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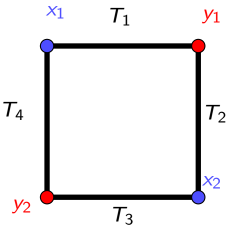
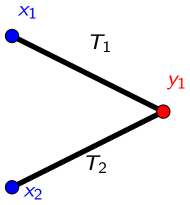
Theorem

Let G be bipartite graph, then $\mathcal{U}$ is given by

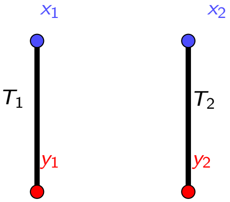
$$\mathcal{U} = \{T_{w^+} - T_{w^-} \mid w \text{ is an even cycle}\}$$
$$\cup \{v_0 T_{w^+} - v_a T_{w^-} \mid w = (v_0, \ldots, v_a) \text{ is an even path}\}$$
$$\cup \{u_0 u_a T_{w_1^+} T_{w_2^-} - v_0 v_b T_{w_1^-} T_{w_2^+} \mid w_1 = (u_0, \ldots, u_a) \text{ and}$$
$$w_2 = (v_0, \ldots, v_b) \text{ are disjoint odd paths}\}.$$



$T_2 T_4 - T_1 T_3$



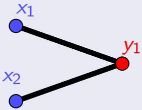
$x_1 T_2 - x_2 T_1$

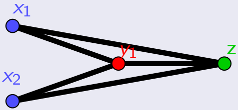


$x_1 y_1 T_2 - x_2 y_2 T_1$

Proof. (sketch).

- We construct the cone graph $C(G)$ of G (add a new vertex z and connect it to all vertices of G).





- Let $\mathbb{K}[C(G)] = \mathbb{K}[e \mid e \in E(C(G))] \subset R[z]$. Then we have a canonical map

$$\pi : S \longrightarrow \mathbb{K}[C(G)] \subset R[z],$$

$$\pi(x_i) = x_i z, \quad \pi(y_i) = y_i z, \quad \pi(T_i) = f_i.$$

  We have that $\mathcal{R}(I) \cong \mathbb{K}[C(G)]$ (Vasconcelos 1998), and so $\mathcal{K} = \mathrm{Ker}(\pi)$.

- From Villarreal 1995 we can determine the circuits of the incidence matrix of $C(G)$. Finally, we translate them into the circuits of M.

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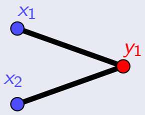

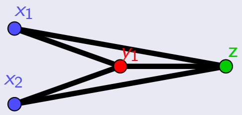

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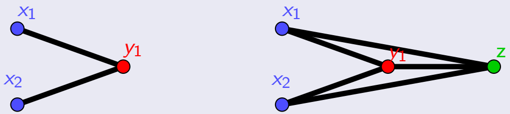

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S is bigraded with $\text{bigrad}(x_i) = \text{bigrad}(y_i) = (1,0)$ and $\text{bigrad}(T_i) = (0,1)$.

$\mathcal{R}(I)$ as a bigraded S-module has a minimal bigraded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I) \longrightarrow 0,$$

where $F_i = \oplus_j S(-a_{ij}, -b_{ij})$. As in Römer 2001, we can define

$$\text{reg}_{xy}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} - i\},$$

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$$\text{reg}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} + b_{ij} - i\}.$$

Theorem (Römer 2001, Chardin 2013)

$$\text{reg}(I^s) \leq 2s + \text{reg}_{xy}(\mathcal{R}(I)) \text{ for all } s \geq 1.$$

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Let $<$ be any term order in S, then we have $\text{reg}_{xy}(\mathcal{R}(I)) \leq \text{reg}_{xy}(S/\text{in}_<(\mathcal{K}))$.

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$$\text{reg}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} + b_{ij} - i\}.$$

Theorem (Römer 2001, Chardin 2013)

$$\text{reg}(I^s) \leq 2s + \text{reg}_{xy}(\mathcal{R}(I)) \text{ for all } s \geq 1.$$

Theorem

Let $<$ be any term order in S, then we have $\text{reg}_{xy}(\mathcal{R}(I)) \leq \text{reg}_{xy}(S/\text{in}_<(\mathcal{K}))$.

S is bigraded with $\text{bigrad}(x_i) = \text{bigrad}(y_i) = (1,0)$ and $\text{bigrad}(T_i) = (0,1)$. $\mathcal{R}(I)$ as a bigraded S-module has a minimal bigraded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I) \longrightarrow 0,$$

where $F_i = \oplus_j S(-a_{ij}, -b_{ij})$. As in Römer 2001, we can define

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Regularity of the powers of I

A celebrated result of Cutkosky, Herzog, and Trung 1999 and Kodiyalam 2000 says that (for a general ideal in a polynomial ring) $\text{reg}(I^s) = as + b$ for $s \gg 0$. But the exact form of this linear function and when $\text{reg}(I^s)$ starts to be linear is still wide open even for monomial ideals.

Corollary

G bipartite graph with bipartition $V(G) = X \cup Y$. Then, for all $s \geq 1$ we have

$$reg(I^s) \leq 2s + \min\{|X|, |Y|\} - 1.$$

Proof.

Using our characterization of $\mathcal{U}$, a "suitable" term order and the Taylor resolution, then we can bound $reg_{xy}(S/in_<(\mathcal{K}))$. □

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Theorem

Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The total regularity of $\mathcal{R}(I)$ is given by

$$reg(\mathcal{R}(I)) = match(G).$$

Proof (sketch).

- *Since M is totally unimodular, then by Gitler, Valencia, and Villarreal 2005 we have that $\mathcal{R}(I)$ is a normal domain.*
- *From Hochster 1972, then $\mathcal{R}(I)$ is Cohen-Macaulay and so it has a canonical module $\omega_{\mathcal{R}(I)}$.*
- *The minimal free resolutions of $\mathcal{R}(I)$ and $\omega_{\mathcal{R}(I)}$ are dual.*
- *$\omega_{\mathcal{R}(I)}$ can be computed using a formula of Danilov and Stanley (Gitler, Valencia, and Villarreal 2005).*

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Corollary

- *For all* $s \geq \text{match}(G) + |E(G)| + 1$ *we have*
$$reg(I(G)^{s+1}) = reg(I(G)^s) + 2.$$
- *For all* $s \geq 1$ *we have*
$$reg(I(G)^s) \leq 2s + match(G) - 1.$$

Proof.

Using the upper bound for the total regularity we get

$$reg_T(\mathcal{R}(I)) \leq match(G),$$
$$reg_{xy}(\mathcal{R}(I)) \leq match(G) - 1.$$

Then the results follow from Cutkosky, Herzog, and Trung 1999 and Römer 2001, respectively. □

A sharper upper bound and a Conjecture

For bipartite graphs, we have the following inequalities

$$\text{reg}(I^s) \leq 2s + \text{co-chord}(G) - 1 \leq 2s + \text{match}(G) - 1 \leq 2s + \min\{|X|, |Y|\} - 1.$$

The upper bound $\text{reg}(I^s) \leq 2s + \text{co-chord}(G) - 1$ was obtained in Jayanthan, Narayanan, and Selvaraja 2016 using a combinatorial argument called "even connection".

Conjecture (Alilooee, Banerjee, Beyarslan and Hà)

Let G be an arbitrary graph then

$$\text{reg}(I(G)^s) \leq 2s + \text{reg}(I(G)) - 2$$

for all $s \geq 1$.

(We always have $2s + \text{co-chord}(G) - 1 \leq 2s + \text{reg}(I(G)) - 2$.)

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References I

Chardin, Marc (2013). “Powers of ideals: Betti numbers, cohomology and regularity”. In: *Commutative algebra*. Springer, New York, pp. 317–333. URL: https://doi.org/10.1007/978-1-4614-5292-8_9.

Cutkosky, S. Dale, Jürgen Herzog, and Ngô Viêt Trung (1999). “Asymptotic behaviour of the Castelnuovo-Mumford regularity”. In: *Compositio Math.* 118.3, pp. 243–261.

Gitler, Isidoro, Carlos Valencia, and Rafael H. Villarreal (2005). “A note on the Rees algebra of a bipartite graph”. In: *J. Pure Appl. Algebra* 201.1-3, pp. 17–24.

Hochster, M. (1972). “Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes”. In: *Ann. of Math. (2)* 96, pp. 318–337.

Jayanthan, AV, N Narayanan, and S Selvaraja (2016). “Regularity of powers of bipartite graphs”. In: *Journal of Algebraic Combinatorics*, pp. 1–22.

References II

Kodiyalam, Vijay (2000). "Asymptotic behaviour of Castelnuovo-Mumford regularity". In: *Proc. Amer. Math. Soc.* 128.2, pp. 407–411.

Römer, Tim (2001). "Homological properties of bigraded algebras". In: *Illinois J. Math.* 45.4, pp. 1361–1376.

Sturmfels, Bernd (1996). *Gröbner bases and convex polytopes*. Vol. 8. University Lecture Series. American Mathematical Society, Providence, RI, pp. xii+162. ISBN: 0-8218-0487-1.

Vasconcelos, Wolmer V. (1998). *Computational methods in commutative algebra and algebraic geometry*. Vol. 2. Algorithms and Computation in Mathematics. Springer-Verlag, Berlin.

Villarreal, Rafael H. (1995). "Rees algebras of edge ideals". In: *Comm. Algebra* 23.9, pp. 3513–3524.

Merci beaucoup!