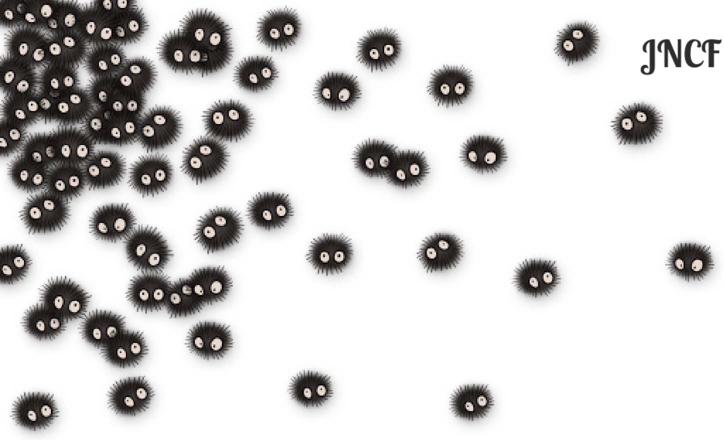
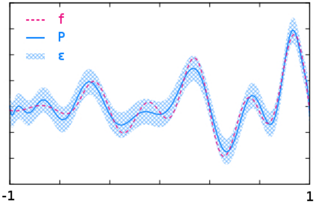


JNCF 2018, *Marseille*

Multinorm Validation and Vector-Valued D-Finite Functions

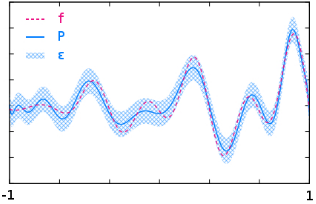
F. Bréhard

- Rigorous Polynomial Approximation = Polynomial + error bound



- Rigorous methods
- Algorithmic methods
- Efficient and accurate
- To be integrated in a large-scale library

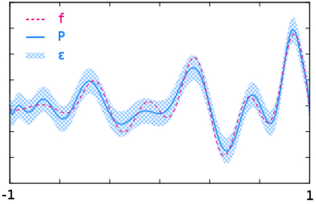
- Rigorous Polynomial Approximation = Polynomial + error bound



- Rigorous methods
- Algorithmic methods
- Efficient and accurate
- To be integrated in a large-scale library

- Solutions of **coupled** systems of linear ordinary differential equations.
- with **componentwise** error bounds.

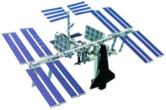
- Rigorous Polynomial Approximation = Polynomial + error bound



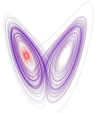
- Rigorous methods
- Algorithmic methods
- Efficient and accurate
- To be integrated in a large-scale library

- Solutions of **coupled** systems of linear ordinary differential equations.
- with **componentwise** error bounds.

- Various fields of applications:

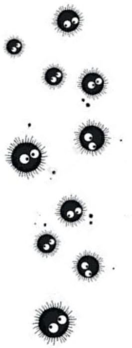


Safety-critical engineering

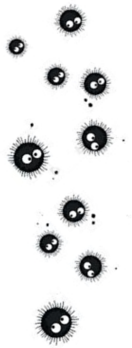


Computer-aided mathematics

Outline



Outline



General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x) + d(\mathbf{T} \cdot x, x^\star)$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x) + \mu d(x, x^\star)$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $(1 - \mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $(1 - \mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}$.

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $(1 - \mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}$.
- $d(x, \mathbf{T} \cdot x) \leqslant d(x, x^\star) + d(x^\star, \mathbf{T} \cdot x)$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1+\mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1-\mu}.$$

Sketch of the proof:

- $(1-\mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1-\mu}$.
- $(1+\mu) d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $(1 - \mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}$.
- $(1 + \mu) d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^*) \geqslant \frac{d(x, \mathbf{T} \cdot x)}{1 + \mu}$.

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1+\mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1-\mu}.$$

Sketch of the proof:

- $(1-\mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1-\mu}$.
- $(1+\mu) d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^*) \geqslant \frac{d(x, \mathbf{T} \cdot x)}{1+\mu}$.

Quasi-Newton Method for $\mathbf{F} \cdot x = 0$

Compute $\mathbf{A} \approx (\mathrm{D}\mathbf{F})_x^{-1}$ in order to define:

$$\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x.$$

Banach fixed-point theorem applies if for some $r > 0$:

- $\mu = \sup_{\tilde{x} \in B(x,r)} \|\mathbf{1} - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_{\tilde{x}}\| < 1$,
- $\|x - \mathbf{T} \cdot x\| + \mu r < r$.

General scheme

- Fixed-point equation $\mathbf{T} \cdot x = x$ with $\mathbf{T}$ contracting,
- Approximation x to exact solution $x^\star$,
- Compute *a posteriori* error bounds with Banach theorem.

Banach Fixed-Point Theorem

If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\mu < 1$,

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\frac{d(x, \mathbf{T} \cdot x)}{1 + \mu} \leqslant d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}.$$

Sketch of the proof:

- $(1 - \mu) d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \leqslant \frac{d(x, \mathbf{T} \cdot x)}{1 - \mu}$.
- $(1 + \mu) d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$
  $\Rightarrow d(x, x^\star) \geqslant \frac{d(x, \mathbf{T} \cdot x)}{1 + \mu}$.

Quasi-Newton Method for $\mathbf{F} \cdot x = 0$

Compute $\mathbf{A} \approx (\mathrm{D}\mathbf{F})_x^{-1}$ in order to define:

$$\mathbf{T} \cdot x = x - \mathbf{A} \cdot \mathbf{F} \cdot x.$$

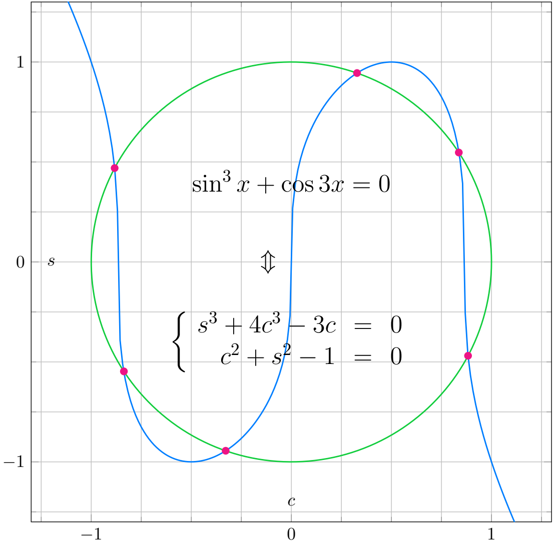
Banach fixed-point theorem applies if for some $r > 0$:

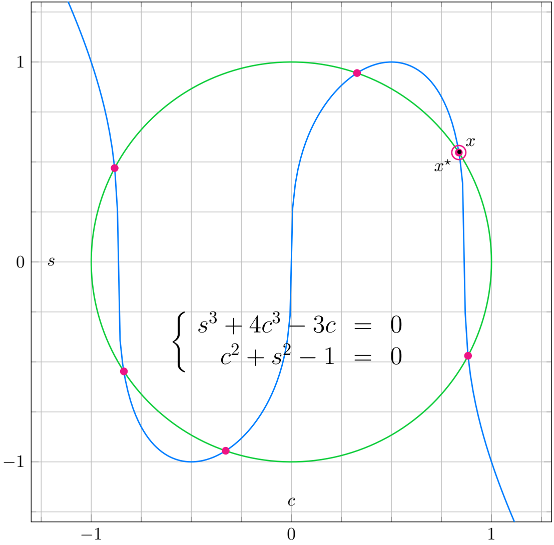
- $\mu = \sup_{\tilde{x} \in B(x,r)} \|\mathbf{1} - \mathbf{A} \cdot \mathrm{D}\mathbf{F}_{\tilde{x}}\| < 1$,
- $\|x - \mathbf{T} \cdot x\| + \mu r < r$.

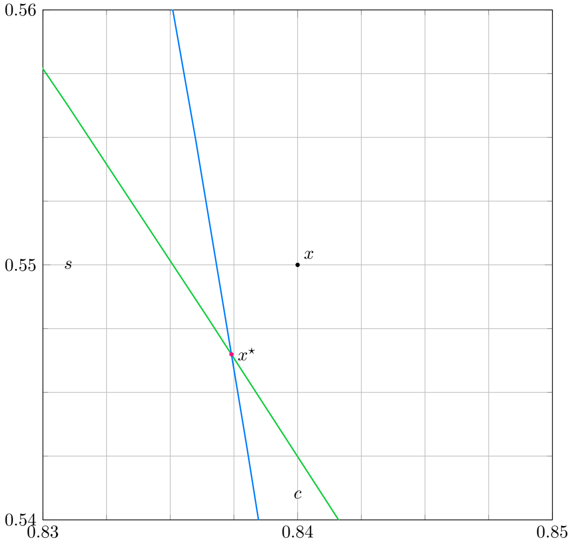
Applications to function space problems:

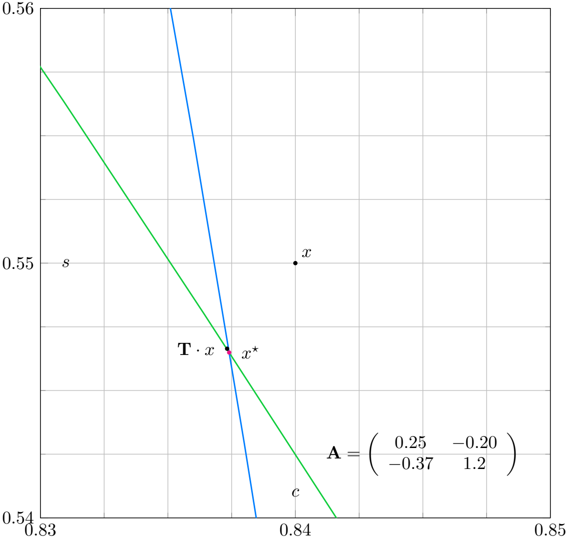
- Early works by Kaucher, Miranker, Yamamoto *et al* (~80's, ~90's).
- Lessard *et al* (2007 - today).
- Benoit, Joldes, Mezzarobba (2011)
  Bréhard, Brisebarre, Joldes (2017).

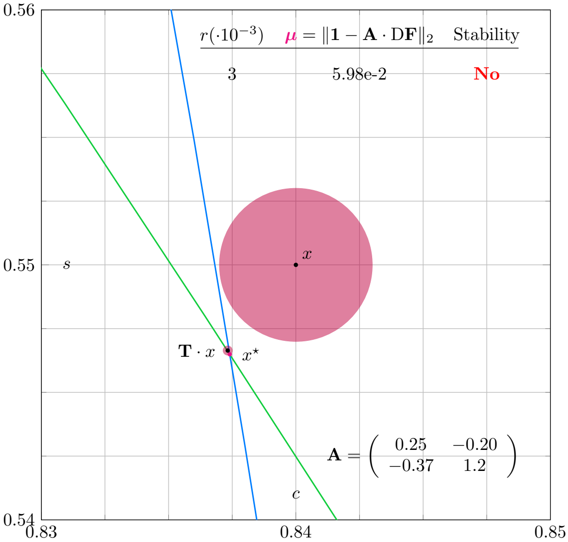
Example: Polynomial Equation in the Plane

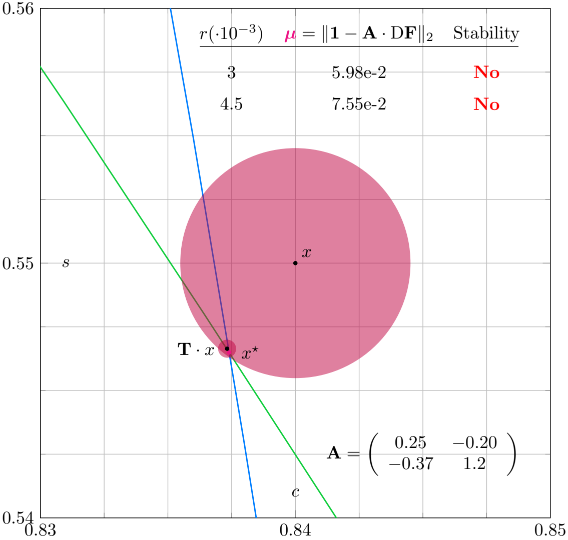



$$\begin{cases} s^3 + 4c^3 - 3c = 0 \\ c^2 + s^2 - 1 = 0 \end{cases}$$
x
$x^\star$
s
c
1
0
−1
−1
0
1


0.56
0.55
0.54
0.83
0.84
0.85
s
x
x*
c

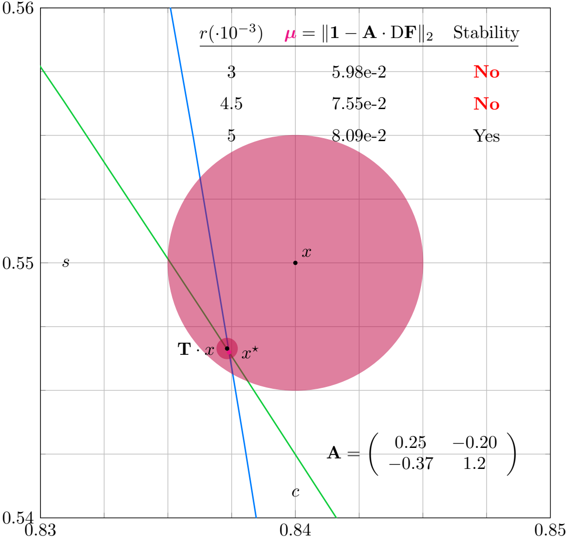

0.56
0.55
0.54
0.83
0.84
0.85
s
x
T · x
x⋆
c
A = (0.25 −0.20 ; −0.37 1.2)


0.56
0.55
0.54
0.83
0.84
0.85
$r(\cdot 10^{-3})$
$\mu = \|\mathbf{1} - \mathbf{A} \cdot \mathrm{D}\mathbf{F}\|_2$
Stability
3
5.98e-2
No
s
x
$\mathbf{T} \cdot x$
$x^\star$
$\mathbf{A} = \begin{pmatrix} 0.25 & -0.20 \\ -0.37 & 1.2 \end{pmatrix}$
c



$r(\cdot 10^{-3})$	$\mu = \|\mathbf{1} - \mathbf{A} \cdot \mathbf{DF}\|_2$	Stability
3	5.98e-2	**No**
4.5	7.55e-2	**No**

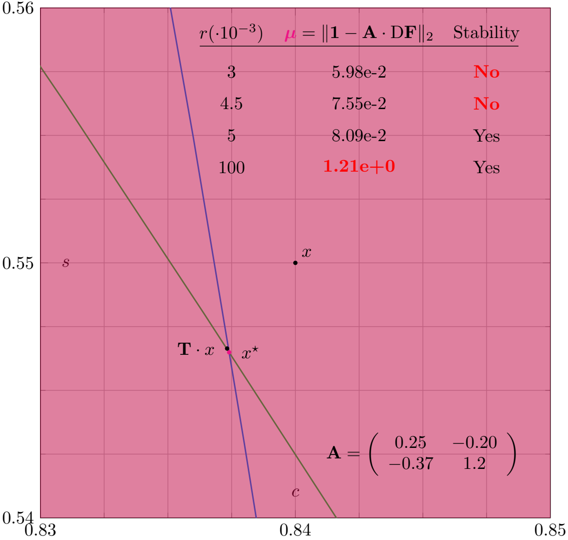
$$\mathbf{A} = \begin{pmatrix} 0.25 & -0.20 \\ -0.37 & 1.2 \end{pmatrix}$$



$r(\cdot 10^{-3})$	$\mu = \lVert \mathbf{1} - \mathbf{A}\cdot \mathbf{DF} \rVert_2$	Stability
3	5.98e-2	**No**
4.5	7.55e-2	**No**
5	8.09e-2	Yes

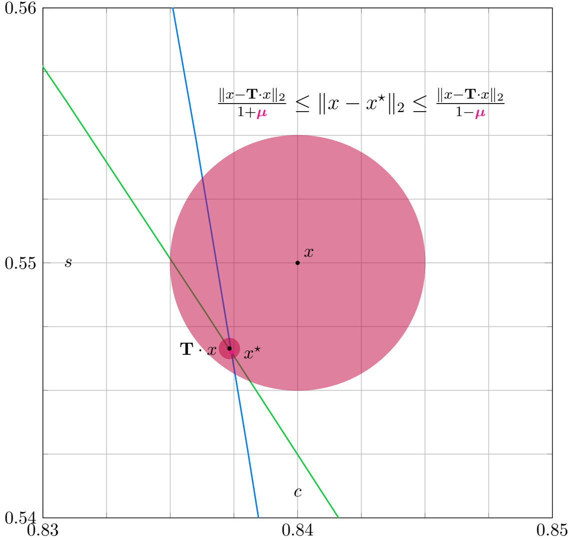
$$\mathbf{A} = \begin{pmatrix} 0.25 & -0.20 \\ -0.37 & 1.2 \end{pmatrix}$$

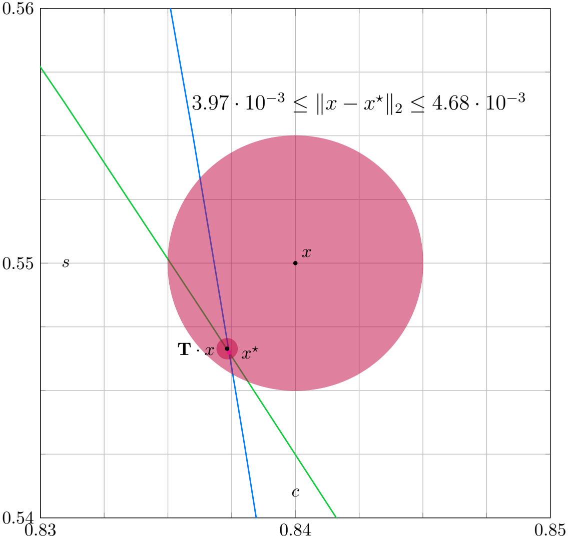
Example: Polynomial Equation in the Plane



$r(\cdot 10^{-3})$	$\mu = \|\mathbf{1} - \mathbf{A}\cdot\mathbf{DF}\|_2$	Stability
3	5.98e-2	**No**
4.5	7.55e-2	**No**
5	8.09e-2	Yes
100	**1.21e+0**	Yes

$$\mathbf{A} = \begin{pmatrix} 0.25 & -0.20 \\ -0.37 & 1.2 \end{pmatrix}$$


$\frac{\|x-\mathbf{T}\cdot x\|_2}{1+\mu} \leq \|x - x^\star\|_2 \leq \frac{\|x-\mathbf{T}\cdot x\|_2}{1-\mu}$
0.56
0.55
0.54
0.83
0.84
0.85
s
x
$\mathbf{T}\cdot x$
$x^\star$
c


0.56
0.55
0.54
0.83
0.84
0.85
$3.97 \cdot 10^{-3} \leq \|x - x^{\star}\|_2 \leq 4.68 \cdot 10^{-3}$
s
x
$\mathbf{T} \cdot x$
$x^{\star}$
c

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \ldots, d_p(x_p, y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\boldsymbol{\Lambda}$-Lipschitz for $\boldsymbol{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \boldsymbol{\Lambda} \cdot d(x, y) \quad \forall x, y \in X$$

Vector-valued Metric and Perov Theorem

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x,y) = (d_1(x_1,y_1), \dots, d_p(x_p,y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \mathbf{\Lambda} \cdot d(x,y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \ldots, d_p(x_p, y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \mathbf{\Lambda} \cdot d(x, y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

$$d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x) + d(\mathbf{T} \cdot x, x^\star)$$

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \dots, d_p(x_p, y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \mathbf{\Lambda} \cdot d(x, y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

$$d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x) + \mathbf{\Lambda} \cdot d(x, x^\star)$$

Vector-Valued Metric

$(X_i, d_i)_{1\leqslant i\leqslant p}$ complete metric spaces.

- $d(x,y) = (d_1(x_1,y_1), \dots, d_p(x_p,y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p\times p}$:

$$d(\mathbf{F}\cdot x, \mathbf{F}\cdot y) \leqslant \mathbf{\Lambda} \cdot d(x,y) \quad \forall x,y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p\times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

$$(\mathbf{1} - \mathbf{\Lambda}) \cdot d(x, x^\star) \leqslant d(x, \mathbf{T}\cdot x)$$

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x, y) = (d_1(x_1, y_1), \ldots, d_p(x_p, y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \mathbf{\Lambda} \cdot d(x, y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

$$(\mathbf{1} - \mathbf{\Lambda}) \cdot d(x, x^\star) \leqslant d(x, \mathbf{T} \cdot x)$$

- $(\mathbf{1} - \mathbf{\Lambda})^{-1} = \mathbf{1} + \mathbf{\Lambda} + \mathbf{\Lambda}^2 + \cdots + \mathbf{\Lambda}^k + \ldots \geqslant \mathbf{0}$.

$\Rightarrow d(x, x^\star) \leqslant (\mathbf{1} - \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)$.

Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x,y) = (d_1(x_1,y_1), \dots, d_p(x_p,y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F}\cdot x, \mathbf{F}\cdot y) \leqslant \mathbf{\Lambda} \cdot d(x,y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

Perov Fixed-Point Theorem

If (X, d) complete vector-valued metric space and $\mathbf{T}$ $\mathbf{\Lambda}$-Lipschitz with $\rho(\mathbf{\Lambda}) < 1$:

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\underbrace{d(x, x^\star)}_{\varepsilon} \leqslant (1 - \mathbf{\Lambda})^{-1} \cdot \underbrace{d(x, \mathbf{T}\cdot x)}_{\eta}.$$

Vector-Valued Metric

$(X_i, d_i)_{1\leqslant i\leqslant p}$ complete metric spaces.

- $d(x,y) = (d_1(x_1,y_1),\dots,d_p(x_p,y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda} \in \mathbb{R}_+^{p\times p}$:

$$d(\mathbf{F}\cdot x, \mathbf{F}\cdot y) \leqslant \mathbf{\Lambda}\cdot d(x,y) \quad \forall x,y \in X$$

Convergent to Zero Matrices

$\mathbf{\Lambda} \in \mathbb{R}^{p\times p}$ is convergent to zero if:

- $\mathbf{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\mathbf{\Lambda}) < 1$.

Generalized Contractions

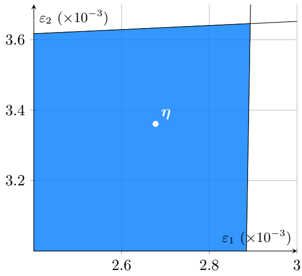
$f : X \to X$ is a *generalized contraction* if it is $\mathbf{\Lambda}$-Lipschitz for $\mathbf{\Lambda}$ convergent to zero.

Perov Fixed-Point Theorem

If (X,d) complete vector-valued metric space and $\mathbf{T}$ $\mathbf{\Lambda}$-Lipschitz with $\rho(\mathbf{\Lambda}) < 1$:

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\underbrace{d(x,x^\star)}_{\varepsilon} \leqslant (1-\mathbf{\Lambda})^{-1}\cdot \underbrace{d(x,\mathbf{T}\cdot x)}_{\eta}.$$



Vector-Valued Metric

$(X_i, d_i)_{1 \leqslant i \leqslant p}$ complete metric spaces.

- $d(x,y) = (d_1(x_1,y_1), \ldots, d_p(x_p,y_p)) \in \mathbb{R}_+^p$ *vector-valued* metric.
- $\mathbf{F} : X \to X$ is $\boldsymbol{\Lambda}$-Lipschitz for $\boldsymbol{\Lambda} \in \mathbb{R}_+^{p \times p}$:

$$d(\mathbf{F} \cdot x, \mathbf{F} \cdot y) \leqslant \boldsymbol{\Lambda} \cdot d(x,y) \quad \forall x, y \in X$$

Convergent to Zero Matrices

$\boldsymbol{\Lambda} \in \mathbb{R}^{p \times p}$ is convergent to zero if:

- $\boldsymbol{\Lambda}^k \to 0$ as $k \to \infty$,
- $\Leftrightarrow \rho(\boldsymbol{\Lambda}) < 1$.

Generalized Contractions

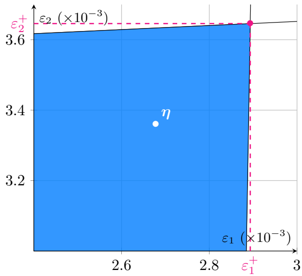
$f : X \to X$ is a *generalized contraction* if it is $\boldsymbol{\Lambda}$-Lipschitz for $\boldsymbol{\Lambda}$ convergent to zero.

Perov Fixed-Point Theorem

If (X, d) complete vector-valued metric space and $\mathbf{T}$ $\boldsymbol{\Lambda}$-Lipschitz with $\rho(\boldsymbol{\Lambda}) < 1$:

- $\mathbf{T}$ admits a unique fixed-point $x^\star$, and
- For all $x \in X$,

$$\underbrace{d(x, x^\star)}_{\varepsilon} \leqslant (1 - \boldsymbol{\Lambda})^{-1} \cdot \underbrace{d(x, \mathbf{T} \cdot x)}_{\eta}.$$



Extending Perov Theorem with Lower Bounds

$$(\mathbf{1} + \mathbf{\Lambda}) \cdot d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$$

$(\mathbf{1} + \mathbf{\Lambda}) \cdot d(x, x^\star) \geqslant d(x, \mathbf{T} \cdot x)$

- $(\mathbf{1} + \mathbf{\Lambda})^{-1} = \mathbf{1} - \mathbf{\Lambda} + \mathbf{\Lambda}^2 - \cdots + (-1)^k \mathbf{\Lambda}^k + \cdots \ngeq \mathbf{0}.$

$\Rightarrow$ Cannot deduce lower bounds!

Error Polytope

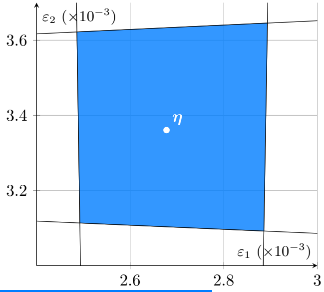
Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

$$\begin{aligned} (\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon &\leqslant \eta \\ (\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon &\geqslant \eta \\ \varepsilon &\geqslant 0 \end{aligned} \tag{P}$$

Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

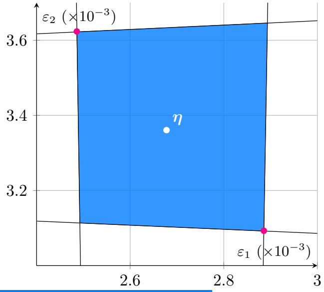
$$\begin{aligned}(\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon &\leqslant \eta \\ (\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon &\geqslant \eta \\ \varepsilon &\geqslant 0\end{aligned} \qquad \text{(P)}$$



Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

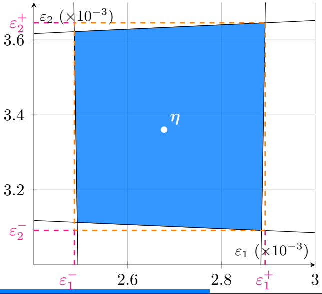
$$\begin{aligned}(\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon &\leqslant \eta \\ (\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon &\geqslant \eta \\ \varepsilon &\geqslant 0\end{aligned} \tag{P}$$



Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

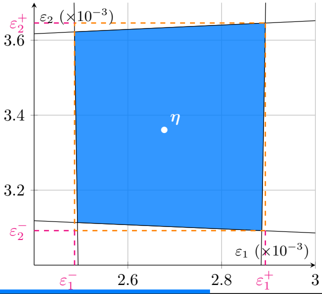
$$\begin{aligned}(\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon &\leqslant \eta \\ (\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon &\geqslant \eta \\ \varepsilon &\geqslant 0\end{aligned} \qquad \text{(P)}$$



Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

$$\begin{aligned}(\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon &\leqslant \eta \\ (\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon &\geqslant \eta \\ \varepsilon &\geqslant 0\end{aligned} \tag{P}$$



Lower Bounds for Perov Theorem

For all $i \in [\![1, p]\!]$,

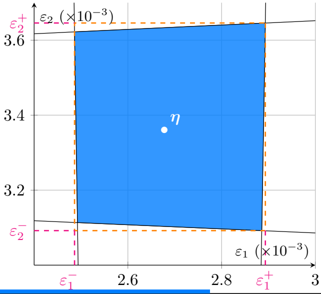
$$d(x, x^\star)_i = \varepsilon_i \geqslant \varepsilon_i^-$$

with ε_i^- given by the intersection of the i-th lower bound constraint together with all the j-th upper bound constraints, for $j \neq i$.

Error Polytope

Let $\varepsilon = d(x, x^*)$ and $\eta = d(x, \mathbf{T} \cdot x)$:

$$(\mathbf{1} - \mathbf{\Lambda}) \cdot \varepsilon \leqslant \eta \qquad \text{(P)}$$
$$(\mathbf{1} + \mathbf{\Lambda}) \cdot \varepsilon \geqslant \eta$$
$$\varepsilon \geqslant 0$$



Lower Bounds for Perov Theorem

For all $i \in [\![1, p]\!]$,

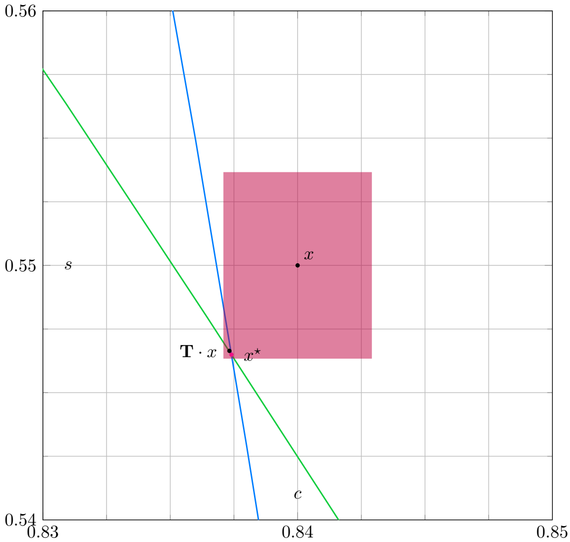
$$d(x, x^\star)_i = \varepsilon_i \geqslant \varepsilon_i^-$$

with ε_i^- given by the intersection of the i-th lower bound constraint together with all the j-th upper bound constraints, for $j \neq i$.

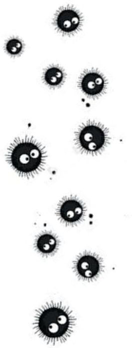
Example

Enclosures obtained by the theorem:

$$\varepsilon_1^- = 2.48 \cdot 10^{-3} \qquad \varepsilon_1^+ = 2.90 \cdot 10^{-3}$$
$$\varepsilon_2^- = 3.09 \cdot 10^{-3} \qquad \varepsilon_2^+ = 3.65 \cdot 10^{-3}$$

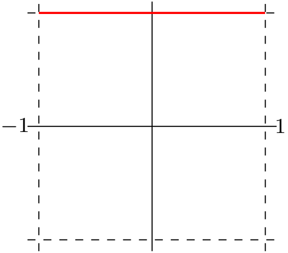

0.56
0.55
0.54
0.83
0.84
0.85
s
x
T · x
x⋆
c

Outline



Chebyshev Family of Polynomials

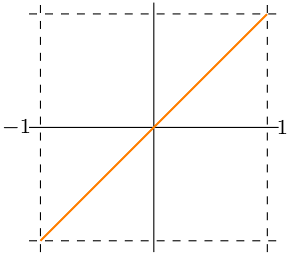
$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$T_0(X) = 1$

Chebyshev Family of Polynomials

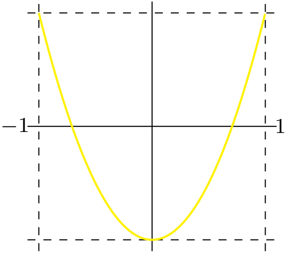
$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$$T_0(X) = 1$$
$$T_1(X) = X$$

Chebyshev Family of Polynomials

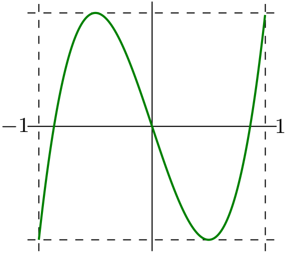
$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$

Chebyshev Family of Polynomials

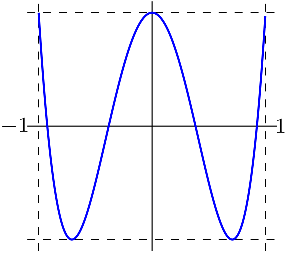
$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$
$$T_3(X) = 4X^3 - 3X$$

Chebyshev Family of Polynomials

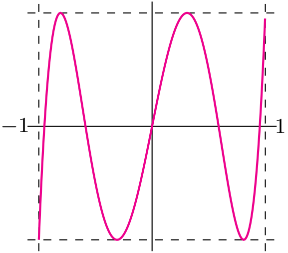
$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$
$$T_3(X) = 4X^3 - 3X$$
$$T_4(X) = 8X^4 - 8X^2 + 1$$

Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$
$$T_3(X) = 4X^3 - 3X$$
$$T_4(X) = 8X^4 - 8X^2 + 1$$
$$T_5(X) = 16X^5 - 20X^3 + 5X$$

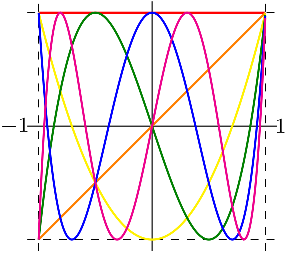
Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

Trigonometric Relation

- $T_n(\cos\vartheta) = \cos n\vartheta.$

$\Rightarrow \forall t \in [-1,1], |T_n(t)| \leqslant 1.$



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$
$$T_3(X) = 4X^3 - 3X$$
$$T_4(X) = 8X^4 - 8X^2 + 1$$
$$T_5(X) = 16X^5 - 20X^3 + 5X$$

Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

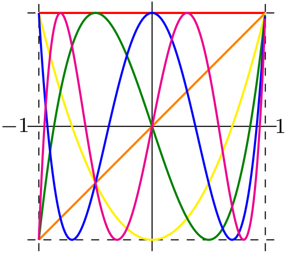
Trigonometric Relation

- $T_n(\cos\vartheta) = \cos n\vartheta$.

$\Rightarrow \forall t \in [-1,1], |T_n(t)| \leqslant 1.$

Multiplication and Integration

- $T_n T_m = \frac{1}{2}(T_{n+m} + T_{n-m})$.
- $\int T_n = \frac{1}{2}\left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1}\right)$.



$$T_0(X) = 1$$
$$T_1(X) = X$$
$$T_2(X) = 2X^2 - 1$$
$$T_3(X) = 4X^3 - 3X$$
$$T_4(X) = 8X^4 - 8X^2 + 1$$
$$T_5(X) = 16X^5 - 20X^3 + 5X$$

Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

Trigonometric Relation

- $T_n(\cos\vartheta) = \cos n\vartheta.$

$\Rightarrow \forall t \in [-1,1], |T_n(t)| \leqslant 1.$

Multiplication and Integration

- $T_n T_m = \frac{1}{2}(T_{n+m} + T_{n-m}).$
- $\int T_n = \frac{1}{2}\left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1}\right).$

Scalar Product and Orthogonality Relations

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\mathrm{d}t = \int_0^{\pi} f(\cos\vartheta)g(\cos\vartheta)\mathrm{d}\vartheta.$$

$\Rightarrow (T_n)_{n\geqslant 0}$ orthogonal family.

Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

Trigonometric Relation

- $T_n(\cos\vartheta) = \cos n\vartheta.$

$\Rightarrow \forall t \in [-1,1], |T_n(t)| \leqslant 1.$

Multiplication and Integration

- $T_n T_m = \frac{1}{2}(T_{n+m} + T_{n-m}).$
- $\int T_n = \frac{1}{2}\left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1}\right).$

Scalar Product and Orthogonality Relations

$$\langle f,g\rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\mathrm{d}t = \int_0^{\pi} f(\cos\vartheta)g(\cos\vartheta)\mathrm{d}\vartheta.$$

$\Rightarrow (T_n)_{n\geqslant 0}$ orthogonal family.

Chebyshev Coefficients and Series

- $a_n = \begin{cases} \frac{2}{\pi}\int_0^{\pi} f(\cos\vartheta)\mathrm{d}\vartheta & n = 0 \\ \frac{1}{\pi}\int_0^{\pi} f(\cos\vartheta)\cos n\vartheta\mathrm{d}\vartheta & n > 0 \end{cases}.$
- $\widehat{f}^{[N]}(t) = \sum_{n\leqslant N} a_n T_n(t), \qquad t \in [-1,1].$

Chebyshev Family of Polynomials

$$T_0(X) = 1,$$
$$T_1(X) = X,$$
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X).$$

Trigonometric Relation

- $T_n(\cos\vartheta) = \cos n\vartheta.$

$\Rightarrow \forall t \in [-1,1], |T_n(t)| \leqslant 1.$

Multiplication and Integration

- $T_n T_m = \frac{1}{2}(T_{n+m} + T_{n-m}).$
- $\int T_n = \frac{1}{2}\left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1}\right).$

Scalar Product and Orthogonality Relations

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}}\mathrm{d}t = \int_0^{\pi} f(\cos\vartheta)g(\cos\vartheta)\mathrm{d}\vartheta.$$

$\Rightarrow (T_n)_{n\geqslant 0}$ orthogonal family.

Chebyshev Coefficients and Series

- $a_n = \begin{cases} \frac{2}{\pi}\int_0^{\pi} f(\cos\vartheta)\mathrm{d}\vartheta & n = 0 \\ \frac{1}{\pi}\int_0^{\pi} f(\cos\vartheta)\cos n\vartheta\mathrm{d}\vartheta & n > 0 \end{cases}.$
- $\widehat{f}^{[N]}(t) = \sum_{n\leqslant N} a_n T_n(t), \qquad t \in [-1,1].$

Convergence Theorems

- If $f \in \mathcal{C}^k$, $\widehat{f}^{[N]} \to f$ in $O(N^{-k})$.
- If f analytic, $\widehat{f}^{[N]} \to f$ exponentially fast.

Vector-Valued D-Finite Equation and Initial Value Problem

$$
\begin{gathered}
Y^{(r)}(t) + A_{r-1}(t) \cdot Y^{(r-1)}(t) + \cdots + A_1(t) \cdot Y'(t) + A_0(t) \cdot Y(t) = G(t) \\
Y(-1) = v_0 \quad Y'(-1) = v_1 \quad \ldots \quad Y^{(r-1)}(-1) = v_{r-1} \quad \in \mathbb{R}^p \\
t \in [-1, 1] \qquad A_i \in \mathbb{R}[t]^{p \times p},\ G \in \mathbb{R}[t]^p.
\end{gathered}
\tag{D}
$$

Vector-Valued D-Finite Equation and Initial Value Problem

$$Y^{(r)}(t) + A_{r-1}(t) \cdot Y^{(r-1)}(t) + \cdots + A_1(t) \cdot Y'(t) + A_0(t) \cdot Y(t) = G(t) \qquad \text{(D)}$$

$$Y(-1) = v_0 \quad Y'(-1) = v_1 \quad \ldots \quad Y^{(r-1)}(-1) = v_{r-1} \quad \in \mathbb{R}^p$$

$$t \in [-1, 1] \qquad A_i \in \mathbb{R}[t]^{p \times p},\ G \in \mathbb{R}[t]^p.$$

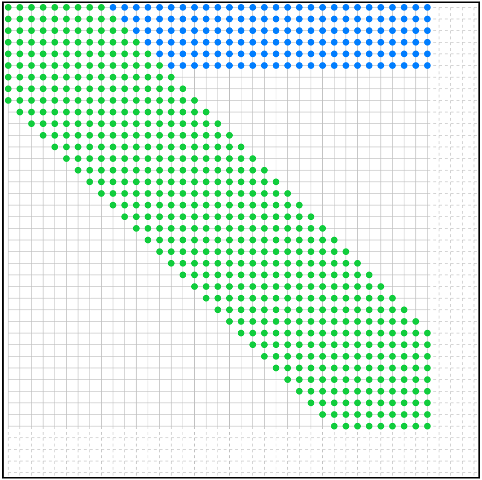
Integral Equation with Polynomial Kernel

(D) becomes:

$$Y(t) + \int_{-1}^{t} \begin{pmatrix} K_{11}(t,s) & \ldots & K_{1p}(t,s) \\ \vdots & \ddots & \vdots \\ K_{p1}(t,s) & \ldots & K_{pp}(t,s) \end{pmatrix} \cdot Y(s)\mathrm{d}s = \Psi(t). \qquad \text{(I)}$$

- $\mathbf{K}_{ij} \cdot y(t) = \int_{-1}^{t} K_{ij}(t,s) y(s) \mathrm{d}s$ 1-dimensional *integral operator*.
- $\mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \ldots & \mathbf{K}_{1p} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{p1} & \ldots & \mathbf{K}_{pp} \end{pmatrix}$ p-dimensional *integral operator*.

$$\mathbf{K}_{ij} \cdot \sum_{k \geqslant 0} c_k T_k \simeq$$

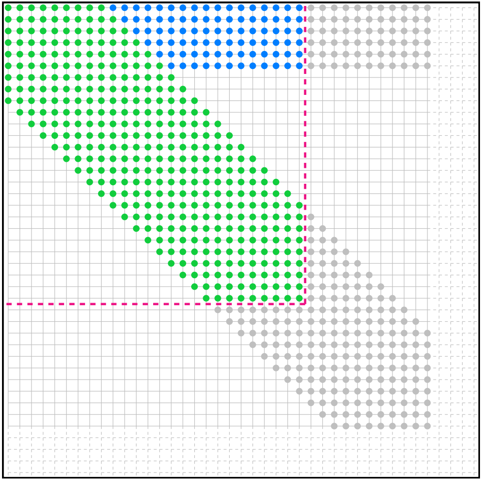


$$\cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_N \\ c_{N+1} \\ \vdots \\ \vdots \end{pmatrix}$$

$\mathbf{K}_{ij}$ is **almost-banded** and **compact**.

Compactness and Almost-Banded Structure of $\mathbf{K}$

$$\mathbf{K}_{ij}^{[N]} \cdot \sum_{k \geqslant 0} c_k T_k \simeq$$

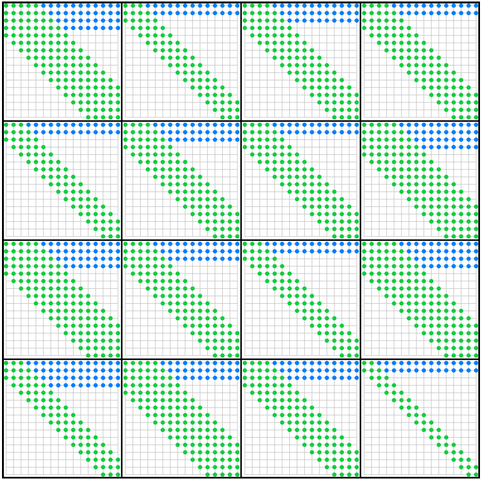


$$\cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \\ 0 \\ \vdots \end{pmatrix}$$

truncated integral operator $\mathbf{K}_{ij}^{[N]}$.

Compactness and Almost-Banded Structure of $\mathbf{K}$

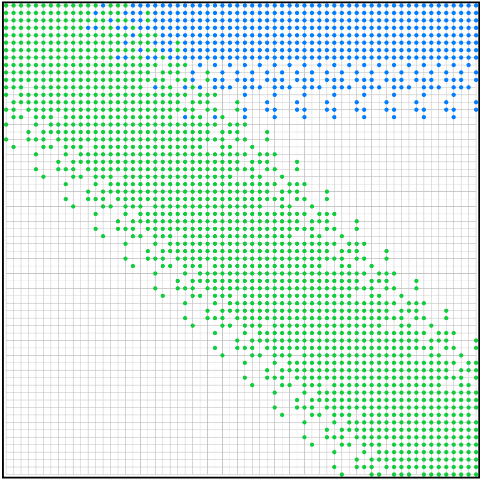
$$\mathbf{K}^{[N]} \cdot \begin{pmatrix} \sum_{k\geqslant 0} c_{1k} T_k \\ \vdots \\ \sum_{k\geqslant 0} c_{pk} T_k \end{pmatrix} \simeq$$



$$\cdot \begin{pmatrix} c_{10} \\ c_{11} \\ \vdots \\ \vdots \\ c_{1N} \\ \hline \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \hline \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \hline c_{p0} \\ c_{p1} \\ \vdots \\ \vdots \\ c_{pN} \end{pmatrix}$$

truncation $\mathbf{K}^{[N]}$ by blocks.

$$\mathbf{K}^{[N]} \cdot \begin{pmatrix} \sum_{k\geqslant 0} c_{1k}\, T_k \\ \vdots \\ \sum_{k\geqslant 0} c_{pk}\, T_k \end{pmatrix} \simeq$$

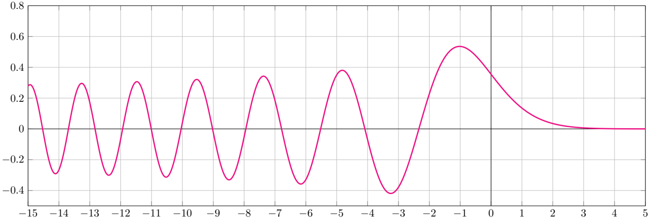


$$\cdot \begin{pmatrix} c_{10} \\ c_{20} \\ \vdots \\ \vdots \\ c_{p0} \\ c_{11} \\ c_{21} \\ \vdots \\ \vdots \\ c_{p1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{1N} \\ c_{2N} \\ \vdots \\ \vdots \\ c_{pN} \end{pmatrix}$$

$\mathbf{K}^{[N]}$ in reordered basis.

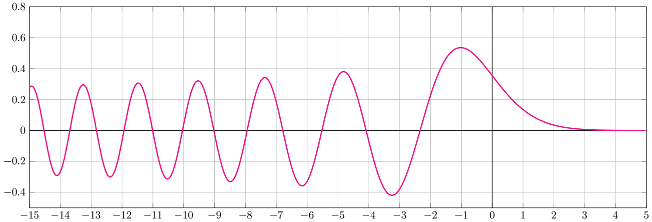
- Airy function Ai defined by:

$$y'' - ty = 0, \qquad \mathrm{Ai}(0) = v_0 \quad \text{and} \quad \mathrm{Ai}'(0) = v_1$$



- Airy function Ai defined by:

$$y'' - ty = 0, \qquad \mathrm{Ai}(0) = v_0 \quad \text{and} \quad \mathrm{Ai}'(0) = v_1$$

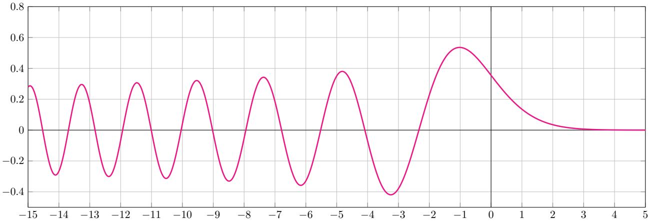


- Integral reformulation over $[-a, 0]$:

$$Y(t) + \int_{-1}^{t} \begin{pmatrix} 0 & -1 \\ s & 0 \end{pmatrix} \cdot Y(s)\mathrm{d}s = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \quad \Rightarrow \quad Y^{\star}(t) = \begin{pmatrix} \mathrm{Ai}(t) \\ \mathrm{Ai}'(t) \end{pmatrix}.$$

- Airy function Ai defined by:

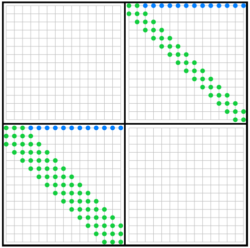
$$y'' - ty = 0, \qquad \mathrm{Ai}(0) = v_0 \quad \text{and} \quad \mathrm{Ai}'(0) = v_1$$



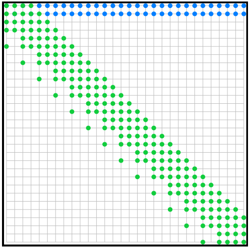
- Integral reformulation over $[-a, 0] \Rightarrow [-1, 1]$:

$$Y(t) + \int_{-1}^{t} \begin{pmatrix} 0 & \frac{a}{2} \\ -\frac{a^2}{4}(s+1) & 0 \end{pmatrix} \cdot Y(s)\,\mathrm{d}s = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \quad \Rightarrow \quad Y^\star(t) = \begin{pmatrix} \mathrm{Ai}(-\frac{a}{2}(t+1)) \\ \mathrm{Ai}'(-\frac{a}{2}(t+1)) \end{pmatrix}.$$

- Truncation at order $N = 14$:

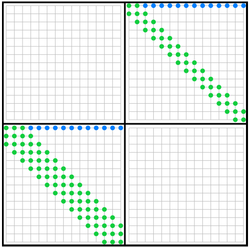


Truncated operator $\mathbf{K}^{[N]}$ by blocks

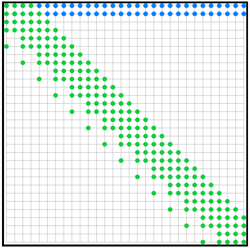


$\mathbf{K}^{[N]}$ in reordered basis

- Truncation at order $N = 14$:



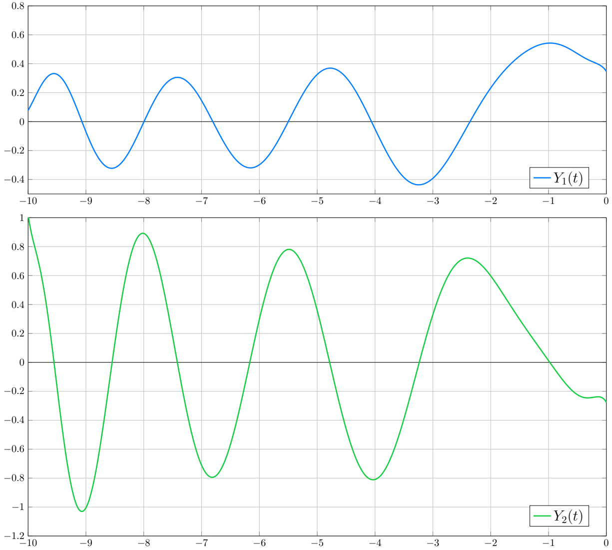
Truncated operator $\mathbf{K}^{[N]}$ by blocks

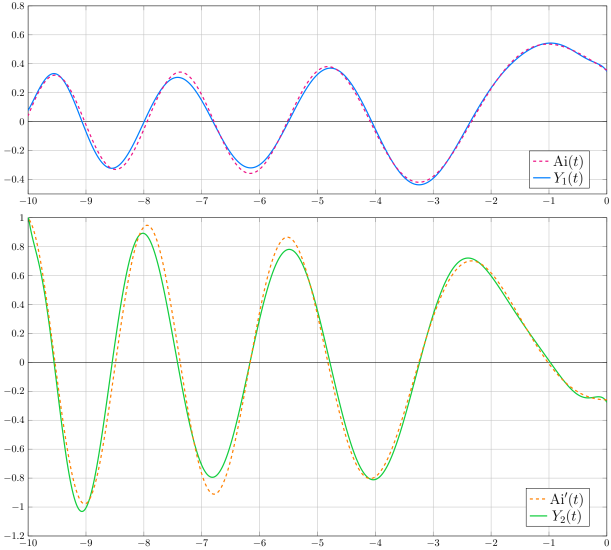


$\mathbf{K}^{[N]}$ in reordered basis

- Obtained approximations for $a = 10$:

$$
\begin{aligned}
Y_1 = {} & +0.139\,T_0 - 0.152\,T_1 + 0.200\,T_2 - 0.016\,T_3 - 0.010\,T_4 + 0.129\,T_5 - 0.112\,T_6 - 0.032\,T_7 \\
& + 0.031\,T_8 - 0.162\,T_9 - 0.111\,T_{10} + 0.103\,T_{11} + 0.110\,T_{12} - 0.005\,T_{13} - 0.033\,T_{14} \\
Y_2 = {} & +0.057\,T_0 + 0.130\,T_1 + 0.052\,T_2 + 0.290\,T_3 + 0.033\,T_4 + 0.273\,T_5 + 0.291\,T_6 + 0.004\,T_7 \\
& + 0.203\,T_8 + 0.104\,T_9 - 0.380\,T_{10} - 0.340\,T_{11} + 0.073\,T_{12} + 0.187\,T_{13} + 0.044\,T_{14}
\end{aligned}
$$


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0
$Y_1(t)$
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0
$Y_2(t)$


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
Ai(t)
Y1(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
Ai′(t)
Y2(t)

Construct $\mathbf{T}$

- Truncation order N_v.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_v]})^{-1}$$

Construct $\mathbf{T}$

- Truncation order N_v.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_v]})^{-1}$$

Ч^1 Banach Space

- $\|y\|_{\text{Ч}^1} = \sum_{n\geqslant 0} |[y]_n| \geqslant \|y\|_\infty$.
- $\|\mathbf{F}\|_{\text{Ч}^1} = \sup_{n\geqslant 0} \|\mathbf{F}\cdot T_n\|_{\text{Ч}^1}$ for $\mathbf{F} : \text{Ч}^1 \to \text{Ч}^1$.

Construct $\mathbf{T}$

- Truncation order N_v.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_v]})^{-1}$$

Ч^1 Banach Space

- $\|y\|_{\text{Ч}^1} = \sum_{n \geqslant 0} |[y]_n| \geqslant \|y\|_\infty$.
- $\|\mathbf{F}\|_{\text{Ч}^1} = \sup_{n \geqslant 0} \|\mathbf{F} \cdot T_n\|_{\text{Ч}^1}$ for $\mathbf{F} : \text{Ч}^1 \to \text{Ч}^1$.
- $\|Y\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^p$ for $Y \in (\text{Ч}^1)^p$.
- $\|\mathbf{F}\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^{p \times p}$ for $\mathbf{F} : (\text{Ч}^1)^p \to (\text{Ч}^1)^p$.

Designing the Newton-like Operator $\mathbf{T}$

Construct $\mathbf{T}$

- Truncation order N_ν.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_\nu]})^{-1}$$

Decomposition of the Operator Norm

$$\|\mathrm{D}\mathbf{T}\|_{(\text{Ч}^1)^p} = \|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K})\|_{(\text{Ч}^1)^p}$$

$$\leqslant \underbrace{\|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K}^{[N_\nu]})\|_{(\text{Ч}^1)^p}}_{\text{Approximation error}} + \underbrace{\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_\nu]})\|_{(\text{Ч}^1)^p}}_{\text{Truncation error}}.$$

Ч^1 Banach Space

- $\|y\|_{\text{Ч}^1} = \sum_{n \geqslant 0} |[y]_n| \geqslant \|y\|_\infty$.
- $\|\mathbf{F}\|_{\text{Ч}^1} = \sup_{n \geqslant 0} \|\mathbf{F} \cdot T_n\|_{\text{Ч}^1}$ for $\mathbf{F} : \text{Ч}^1 \to \text{Ч}^1$.
- $\|Y\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^p$ for $Y \in (\text{Ч}^1)^p$.
- $\|\mathbf{F}\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^{p \times p}$ for $\mathbf{F} : (\text{Ч}^1)^p \to (\text{Ч}^1)^p$.

Construct $\mathbf{T}$

- Truncation order N_ν.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_\nu]})^{-1}$$

Decomposition of the Operator Norm

$$\|\mathrm{D}\mathbf{T}\|_{(\Psi^1)^p} = \|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K})\|_{(\Psi^1)^p}$$

$$\leqslant \underbrace{\|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K}^{[N_\nu]})\|_{(\Psi^1)^p}}_{\text{Approximation error}} + \underbrace{\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_\nu]})\|_{(\Psi^1)^p}}_{\text{Truncation error}}.$$

Ψ^1 Banach Space

- $\|y\|_{\Psi^1} = \sum_{n\geqslant 0} |[y]_n| \geqslant \|y\|_\infty$.
- $\|\mathbf{F}\|_{\Psi^1} = \sup_{n\geqslant 0} \|\mathbf{F} \cdot T_n\|_{\Psi^1}$ for $\mathbf{F} : \Psi^1 \to \Psi^1$.
- $\|Y\|_{(\Psi^1)^p} \in \mathbb{R}_+^p$ for $Y \in (\Psi^1)^p$.
- $\|\mathbf{F}\|_{(\Psi^1)^p} \in \mathbb{R}_+^{p\times p}$ for $\mathbf{F} : (\Psi^1)^p \to (\Psi^1)^p$.

Approximation error:

- Finite-dimensional problem.
- Matrix multiplications and Ψ^1-norm.

Construct $\mathbf{T}$

- Truncation order N_v.
- Approx inverse:

$$\mathbf{A} \approx (\mathbf{1} + \mathbf{K}^{[N_v]})^{-1}$$

Decomposition of the Operator Norm

$$\|\mathrm{D}\mathbf{T}\|_{(\text{Ч}^1)^p} = \|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K})\|_{(\text{Ч}^1)^p}$$

$$\leqslant \underbrace{\|\mathbf{1} - \mathbf{A} \cdot (\mathbf{1} + \mathbf{K}^{[N_v]})\|_{(\text{Ч}^1)^p}}_{\text{Approximation error}} + \underbrace{\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]})\|_{(\text{Ч}^1)^p}}_{\text{Truncation error}}.$$

Ч^1 Banach Space

- $\|y\|_{\text{Ч}^1} = \sum_{n \geqslant 0} |[y]_n| \geqslant \|y\|_\infty$.
- $\|\mathbf{F}\|_{\text{Ч}^1} = \sup_{n \geqslant 0} \|\mathbf{F} \cdot T_n\|_{\text{Ч}^1}$ for $\mathbf{F} : \text{Ч}^1 \to \text{Ч}^1$.
- $\|Y\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^p$ for $Y \in (\text{Ч}^1)^p$.
- $\|\mathbf{F}\|_{(\text{Ч}^1)^p} \in \mathbb{R}_+^{p \times p}$ for $\mathbf{F} : (\text{Ч}^1)^p \to (\text{Ч}^1)^p$.

Approximation error:

- Finite-dimensional problem.
- Matrix multiplications and Ч^1-norm.

Truncation error:

- Infinite-dimensional problem.
- Crude bounds $\Rightarrow$ large N_v.
- Smart bounding techniques.

Rigorous Chebyshev Approximation - Summary

1. Integral reformulation,
2. Numerical approximation Y of $Y^\star$,
3. Creating Newton-like operator $\mathbf{T}$,
4. Computing $\boldsymbol{\Lambda} \geqslant \|\mathrm{D}\mathbf{T}\|_{(\mathbb{Y}^1)^p}$,
5. If $\rho(\boldsymbol{\Lambda}) < 1$, bound $\|Y - \mathbf{T} \cdot T\|_{(\mathbb{Y}^1)^p}$ and apply Perov theorem.

Rigorous Chebyshev Approximation - Summary

1. Integral reformulation,
2. Numerical approximation Y of $Y^\star$,
3. Creating Newton-like operator $\mathbf{T}$,
4. Computing $\mathbf{\Lambda} \geqslant \|\mathrm{D}\mathbf{T}\|_{(\mathbb{Y}^1)^p}$,
5. If $\rho(\mathbf{\Lambda}) < 1$, bound $\|Y - \mathbf{T} \cdot T\|_{(\mathbb{Y}^1)^p}$ and apply Perov theorem.

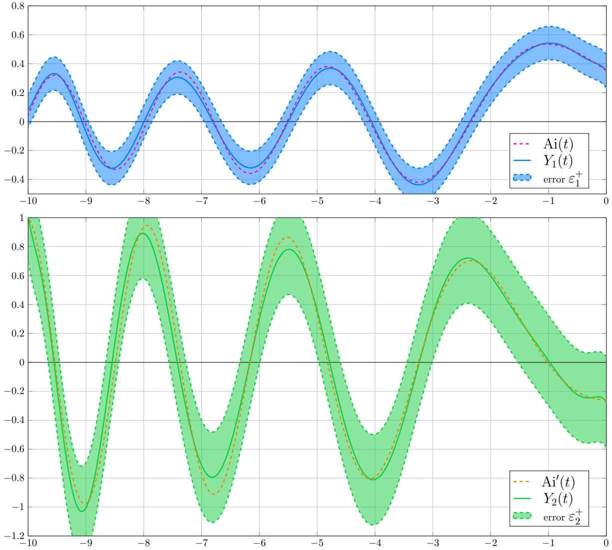
Example: Airy Function over $[-10, 0]$

- with $N_v = 1000$:

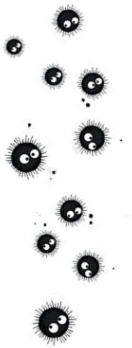
$$\mathbf{\Lambda} = \begin{pmatrix} 7.56 \cdot 10^{-4} & 8.71 \cdot 10^{-3} \\ 3.92 \cdot 10^{-2} & 1.11 \cdot 10^{-2} \end{pmatrix}$$

- $\varepsilon_1^- \leqslant \|Y_1 - \mathrm{Ai}\|_{\mathbb{Y}^1} \leqslant \varepsilon_1^+$ and $\varepsilon_2^- \leqslant \|Y_2 - \mathrm{Ai}'\|_{\mathbb{Y}^1} \leqslant \varepsilon_2^+$ with:

$$\varepsilon_1^- = 0.109 \qquad \varepsilon_1^+ = 0.115$$
$$\varepsilon_2^- = 0.296 \qquad \varepsilon_2^+ = 0.312$$


Ai(t)
Y1(t)
error ε1+
Ai′(t)
Y2(t)
error ε2+

Outline



Conclusion and Future Work

- A general framework for **multinorm validation**.
- An algorithm for **Rigorous Polynomial Approximations** to vector-valued D-finite functions.
- Generalization to **non-polynomial** systems of linear ODEs.
- C library freely available at `https://gforge.inria.fr/projects/tchebyapprox`.
- Towards a **certified Coq** implementation.

Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:

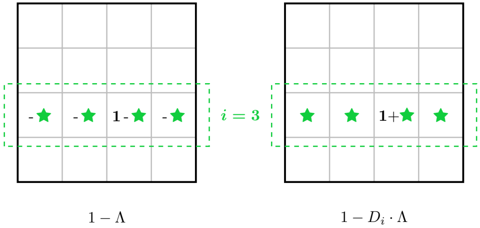
Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\boldsymbol{\Lambda}$-Lipschitz with $\boldsymbol{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x) \right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:

$$d = \det(\mathbf{1} - \boldsymbol{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})$$

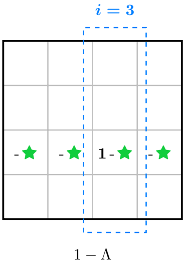


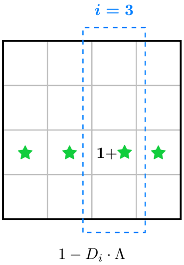
Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\boldsymbol{\Lambda}$-Lipschitz with $\boldsymbol{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:





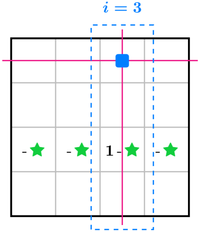
$$d = \det(\mathbf{1} - \boldsymbol{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})$$

Lower Bounds for Perov Theorem

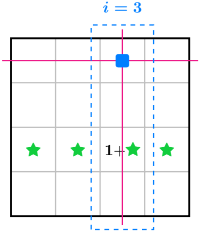
If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:



$1 - \Lambda$



$1 - D_i \cdot \Lambda$

$$d = \det(\mathbf{1} - \mathbf{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})$$

$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i1} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i1}$$

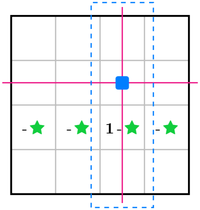
Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

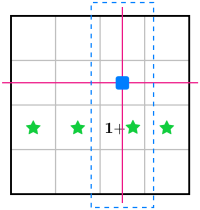
Sketch of the proof:

$i = 3$



$1 - \Lambda$

$i = 3$



$1 - D_i \cdot \Lambda$

$$d = \det(\mathbf{1} - \mathbf{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})$$

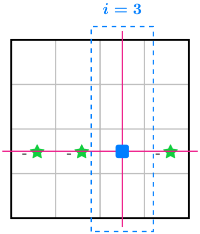
$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i1} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i1}$$
$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i2} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i2}$$

Lower Bounds for Perov Theorem

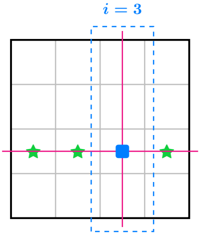
If $\mathbf{T}$ is $\boldsymbol{\Lambda}$-Lipschitz with $\boldsymbol{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:



$1 - \Lambda$



$1 - D_i \cdot \Lambda$

$$d = \det(\mathbf{1} - \boldsymbol{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})$$

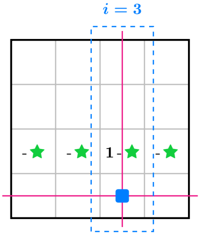
$$d_i (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i1} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i1}$$
$$d_i (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i2} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i2}$$
$$d_i (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i3} = +d_i (\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i3}$$

Lower Bounds for Perov Theorem

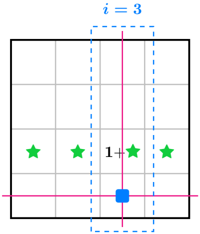
If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:



$1 - \Lambda$



$1 - D_i \cdot \Lambda$

$$d = \det(\mathbf{1} - \mathbf{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})$$

$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i1} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i1}$$
$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i2} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i2}$$
$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i3} = +d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i3}$$
$$d_i (\mathbf{1} - \mathbf{\Lambda})^{-1}_{i4} = -d_i (\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i4}$$

Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:

- $(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{ii}^{-1} \geqslant 0$, and
- $(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{ij}^{-1} \leqslant 0$ for $j \neq i$.

$$d = \det(\mathbf{1} - \mathbf{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})$$

$$d_i(\mathbf{1} - \mathbf{\Lambda})_{i1}^{-1} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{i1}^{-1}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})_{i2}^{-1} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{i2}^{-1}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})_{i3}^{-1} = +d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{i3}^{-1}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})_{i4}^{-1} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})_{i4}^{-1}$$

Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\mathbf{\Lambda}$-Lipschitz with $\mathbf{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \boxed{-1} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:

- $(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{ii} \geqslant 0$, and
- $(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{ij} \leqslant 0$ for $j \neq i$.

$$(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda}) \cdot \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_i \\ \vdots \\ \varepsilon_p \end{pmatrix} \begin{matrix} \leqslant \\ \vdots \\ \geqslant \\ \vdots \\ \leqslant \end{matrix} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_i \\ \vdots \\ \eta_p \end{pmatrix}$$

$$d = \det(\mathbf{1} - \mathbf{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})$$

$$d_i(\mathbf{1} - \mathbf{\Lambda})^{-1}_{i1} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i1}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})^{-1}_{i2} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i2}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})^{-1}_{i3} = +d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i3}$$
$$d_i(\mathbf{1} - \mathbf{\Lambda})^{-1}_{i4} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \mathbf{\Lambda})^{-1}_{i4}$$

Lower Bounds for Perov Theorem

If $\mathbf{T}$ is $\boldsymbol{\Lambda}$-Lipschitz with $\boldsymbol{\Lambda}$ convergent to zero, then for all $i \in [\![1, p]\!]$:

$$d(x, x^\star)_i \geqslant \varepsilon_i^- = \left((\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})^{-1} \cdot d(x, \mathbf{T} \cdot x)\right)_i \qquad \text{with} \quad \mathbf{D}_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{-1} & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Sketch of the proof:

- $(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{ii} \geqslant 0$, and
- $(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{ij} \leqslant 0$ for $j \neq i$.

$$(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda}) \cdot \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_i \\ \vdots \\ \varepsilon_p \end{pmatrix} \begin{matrix} \leqslant \\ \vdots \\ \geqslant \\ \vdots \\ \leqslant \end{matrix} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_i \\ \vdots \\ \eta_p \end{pmatrix}$$

$\Rightarrow \varepsilon_i \geqslant \left((\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1} \cdot \eta\right)_i.$

$$d = \det(\mathbf{1} - \boldsymbol{\Lambda})$$
$$d_i = \det(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})$$

$$d_i(\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i1} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i1}$$
$$d_i(\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i2} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i2}$$
$$d_i(\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i3} = +d_i(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i3}$$
$$d_i(\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{i4} = -d_i(\mathbf{1} - \mathbf{D}_i \cdot \boldsymbol{\Lambda})^{-1}_{i4}$$

Tightness of Error Enclosures [Appendix]

- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i \eta_i + \sum_{j \neq i} c_j \eta_j}{c_i \eta_i - \sum_{j \neq i} c_j \eta_j},$$

$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

Tightness of Error Enclosures [Appendix]

- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \quad \Leftrightarrow$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i \eta_i + \sum_{j \neq i} c_j \eta_j}{c_i \eta_i - \sum_{j \neq i} c_j \eta_j},$$

$c_j = (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \boldsymbol{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \quad \Leftrightarrow$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j \neq i} c_j \eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1 \leqslant i \leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j \neq i} c_j \eta_j \right\}$$

- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

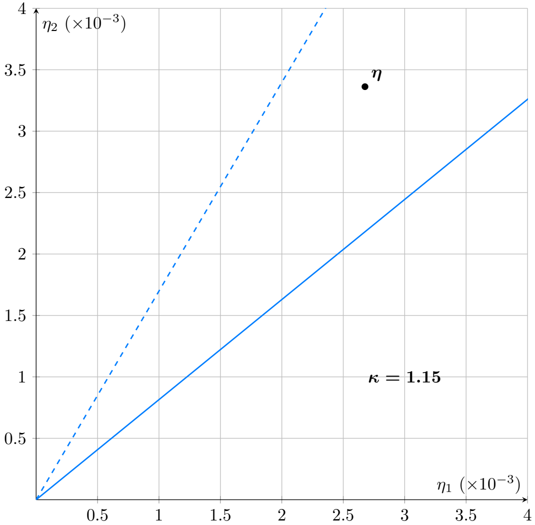
$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \iff$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j \right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

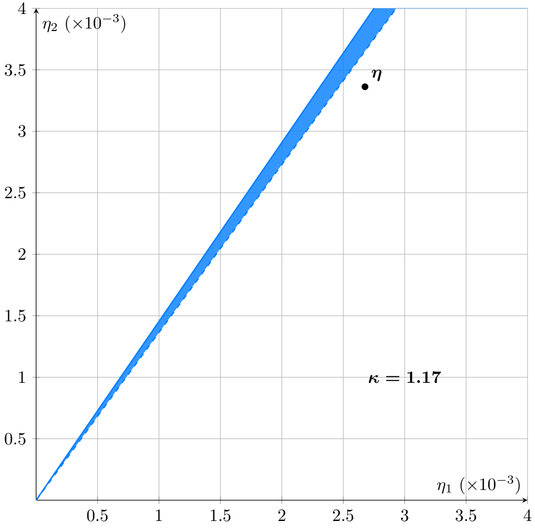
$c_j = (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \boldsymbol{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \iff$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j \right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

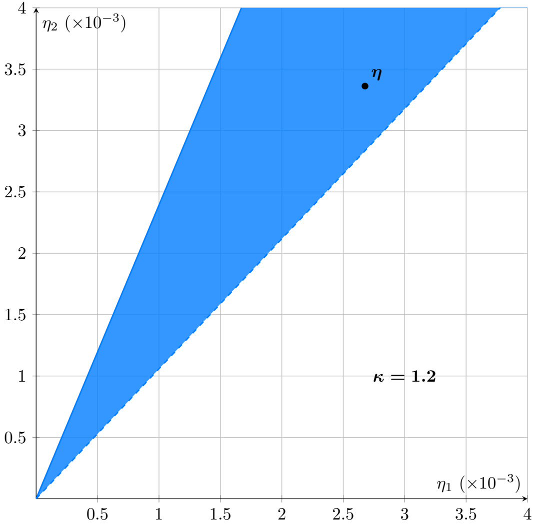
$c_j = (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \boldsymbol{\Lambda})$, $d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \iff$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j \right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d}\,\frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

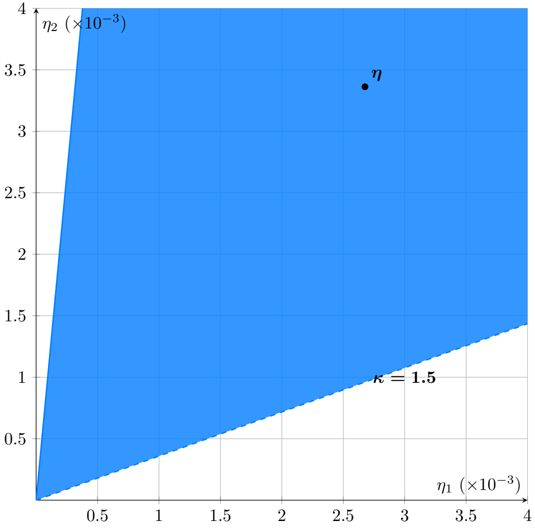
$c_j = (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \boldsymbol{\Lambda})$, $d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \quad \Leftrightarrow$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'}\,\frac{1}{c_i}\sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p}\left\{\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'}\,\frac{1}{c_i}\sum_{j\neq i} c_j\eta_j\right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

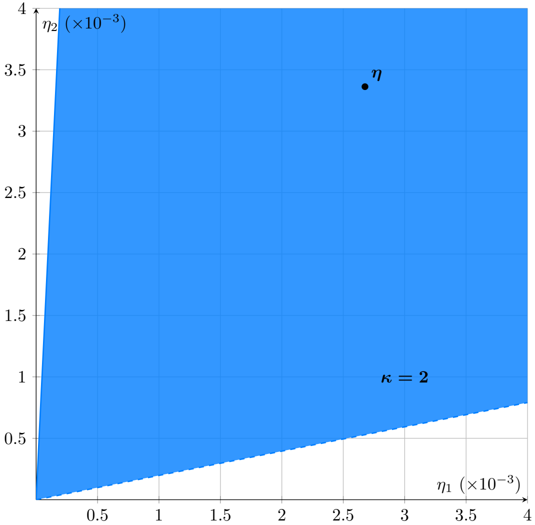
$c_j = (\mathbf{1} - \boldsymbol{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \boldsymbol{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \boldsymbol{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \quad \Leftrightarrow$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j \right\}$$



Tightness of Error Enclosures [Appendix]

- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i \eta_i + \sum_{j \neq i} c_j \eta_j}{c_i \eta_i - \sum_{j \neq i} c_j \eta_j},$$

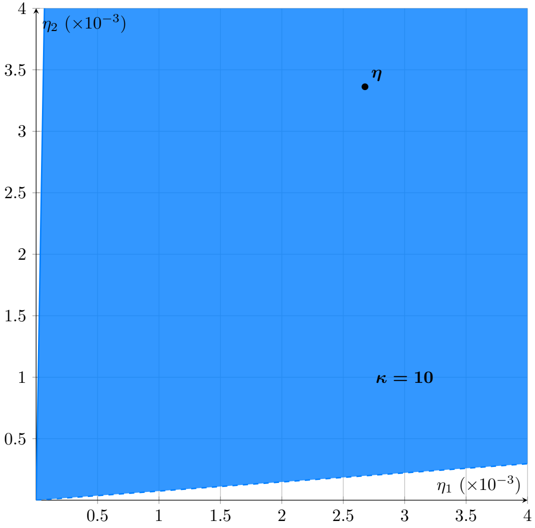
$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \iff$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j \neq i} c_j \eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1 \leqslant i \leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j \neq i} c_j \eta_j \right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d}\,\frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

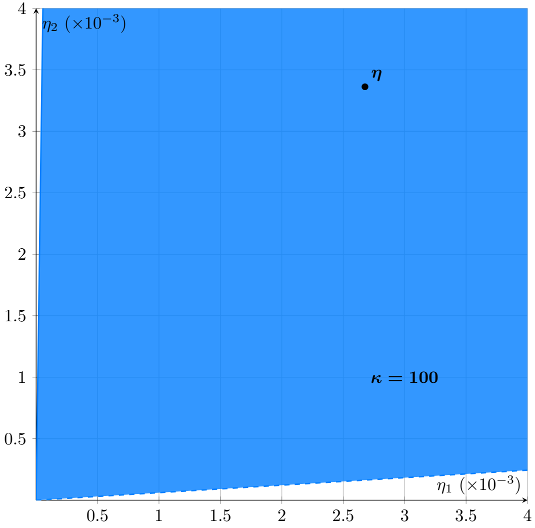
$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \quad \Leftrightarrow$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'}\,\frac{1}{c_i}\sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p}\left\{\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'}\,\frac{1}{c_i}\sum_{j\neq i} c_j\eta_j\right\}$$



- Overapproximation ratio:

$$\frac{\varepsilon_i^+}{\varepsilon_i^-} = \frac{d'}{d} \frac{c_i\eta_i + \sum_{j\neq i} c_j\eta_j}{c_i\eta_i - \sum_{j\neq i} c_j\eta_j},$$

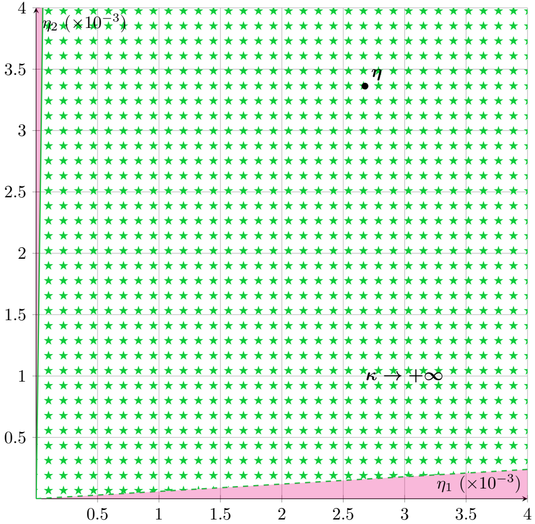
$c_j = (\mathbf{1} - \mathbf{\Lambda})^{-1}_{ij}$, $d = \det(\mathbf{1} - \mathbf{\Lambda})$,
$d' = \det(\mathbf{1} - \mathbf{D_i} \cdot \mathbf{\Lambda})$.

- $\frac{\varepsilon_i^+}{\varepsilon_i^-} \leqslant \kappa \iff$

$$\eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j$$

Tightness Cone

$$\mathcal{C}_\kappa = \bigcap_{1\leqslant i\leqslant p} \left\{ \eta_i \geqslant \frac{\kappa d + d'}{\kappa d - d'} \frac{1}{c_i} \sum_{j\neq i} c_j\eta_j \right\}$$

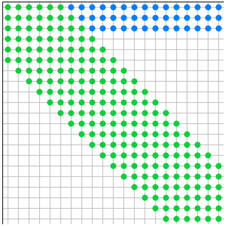


Truncation Error

$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$

Truncation Error

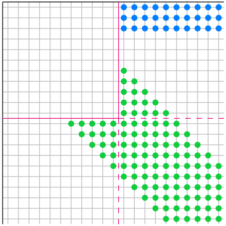
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$



$\mathbf{K}$

Truncation Error

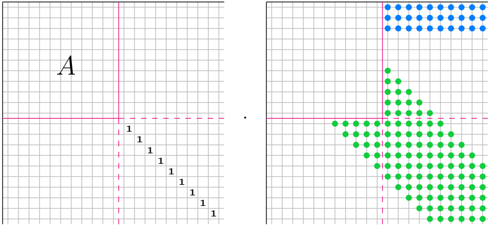
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$



$\mathbf{K}-\mathbf{K}^{[N]}$

Truncation Error

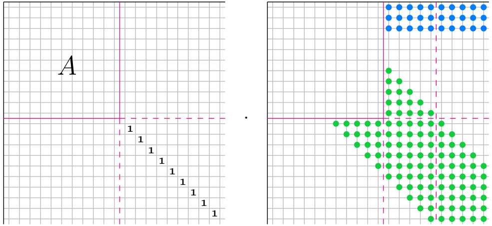
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_\nu]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_\nu]})\cdot T_i\|$$



$\mathbf{A} \quad \cdot \quad \mathbf{K}-\mathbf{K}^{[N]}$

Truncation Error

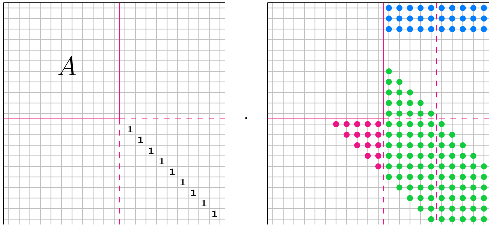
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$



$\mathbf{A} \cdot \mathbf{K}-\mathbf{K}^{[N]}$

Truncation Error

$$\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]})\| = \sup_{i \geqslant 0} \|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]}) \cdot T_i\|$$

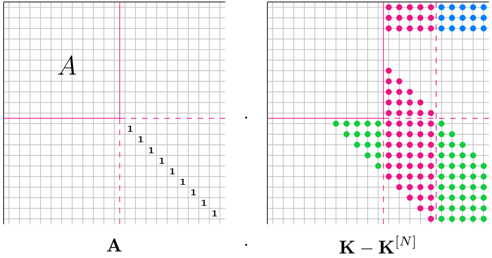


$\mathbf{A} \quad \cdot \quad \mathbf{K} - \mathbf{K}^{[N]}$

1. Direct computation.

Truncation Error

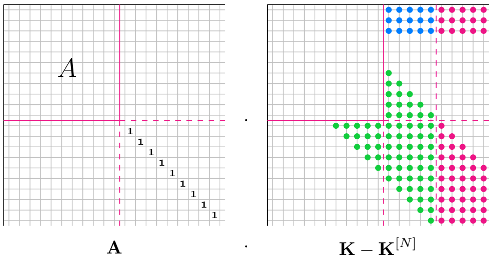
$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$



1. Direct computation.
2. Direct computation.

Truncation Error

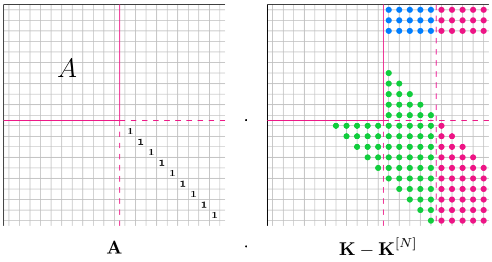
$$\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]})\| = \sup_{i \geqslant 0} \|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]}) \cdot T_i\|$$



1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:

Truncation Error

$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0} \|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$

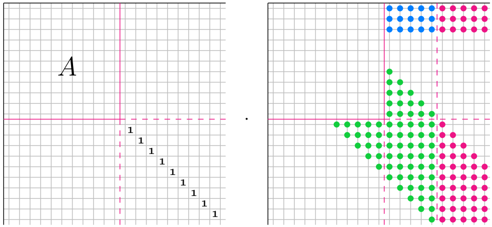


1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in $1/i$ and $1/i^2$: possibly large overestimations.

$$diag(i) \leqslant \frac{C}{i} \qquad init(i) \leqslant \frac{D}{i^2}$$

Truncation Error

$$\|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]})\| = \sup_{i \geqslant 0} \|\mathbf{A} \cdot (\mathbf{K} - \mathbf{K}^{[N_v]}) \cdot T_i\|$$



$\mathbf{A}$ · $\mathbf{K} - \mathbf{K}^{[N]}$

1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in $1/i$ and $1/i^2$: possibly large overestimations.

$$diag(i) \leqslant \frac{C}{i} \qquad init(i) \leqslant \frac{D}{i^2}$$

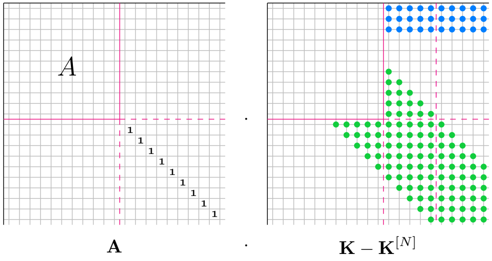
   - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

$$diag(i) \leqslant diag(i_0) + \frac{C'}{i^2}$$

$$init(i) \leqslant init(i_0) + \frac{D'}{i^4}$$

Truncation Error

$$\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\| = \sup_{i\geqslant 0}\|\mathbf{A}\cdot(\mathbf{K}-\mathbf{K}^{[N_v]})\cdot T_i\|$$



1. Direct computation.
2. Direct computation.
3. Bound the remaining *infinite* number of columns:
   - Using the bounds in $1/i$ and $1/i^2$: possibly large overestimations.

$$diag(i) \leqslant \frac{C}{i} \qquad init(i) \leqslant \frac{D}{i^2}$$

   - Using a first order difference method: differences in $1/i^2$ and $1/i^4$.

$$diag(i) \leqslant diag(i_0) + \frac{C'}{i^2}$$

$$init(i) \leqslant init(i_0) + \frac{D'}{i^4}$$