A Symbolic Approach for Solving Algebraic Riccati Equations

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Overview

Overview

The linear optimal control problem

Input : a linear dynamical system

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ x(0) = x_0 \end{cases}$$

where

$x(t) \in \mathbb{R}^n$ the state vector, $u(t) \in \mathbb{R}^m$ the control vector

A (resp. B) is an $n \times n$ (resp. $n \times m$) real matrix

Output : a control u that stabilizes the system and minimizes a quadratic cost functional

$$\frac{1}{2}\int_0^{+\infty} [x(t)^T\,Q\,x(t) + u(t)^T\,R\,u(t)]d\,t$$

where Q (resp. R) is a positive semi-definite (resp. positive definite) symmetric real matrix.

Goal : Achieve a control reference using the minimum energy

Optimal control : mathematical simplifications

Let introduce the Lagrange multiplier $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the following functional

$$\frac{1}{2}\int_0^{+\infty} [x(t)^T \, Q \, x(t) + u(t)^T \, R \, u(t) - \lambda(t)(\dot{x}(t) - A\,x(t) - B\,u(t))] d\,t$$

By a variation computation, the problem is reduced to solving the following OD systems

$$\left\{ \begin{array}{l} \dot{\lambda}(t)^T + A^T \lambda(t)^T + Q\,x(t) = 0, \\ \dot{x}(t) - A\,x(t) - B\,u(t) = 0, \\ R\,u(t) + B^T \lambda(t)^T = 0. \end{array} \right. \xrightarrow{u = -R^{-1} B^T \lambda(t)^T} \left\{ \begin{array}{l} \dot{x}(t) = A\,x(t) - B\,R^{-1} B^T \lambda(t)^T, \\ \dot{\lambda}(t)^T = -Q\,x(t) - A^T \lambda(t)^T. \end{array} \right.$$

If we seek for a solution of the form $\lambda(t)^T = P(t)\,x(t)$, $P(t)$ must satisfy the differential equation

$$\dot{P} = A\,P + A^T\,P + P\,B\,R^{-1}\,B^T\,P^T + Q$$

If we consider a constant matrix P, this yields the following algebraic equation

$$A\,P + A^T\,P + P\,B\,R^{-1}\,B^T\,P^T + Q = 0$$

The optimal control is then given as $u(t) = -R^{-1}\,B^T\,P\,x(t)$

Algebraic Riccati Equations

An **Algebraic Riccati Equation** is the following quadratic matrix equation

$$A^T X + X A + X B R^{-1} B^T X + Q = 0 \tag{1}$$

where A is a real $n \times n$ matrix and Q, R are real symmetric $n \times n$ matrices

Solving Algebraic Riccati Equations

- Computing all the solutions X of (1)
- Computing specific solutions of (1) : real, hermitian, positive definite...
- A positive definite solution is stabilizing

Algebraic Riccati Equations are fundamental in many linear control theory problems (Estimation, Filtering, Robust control,...)

Riccati Equations and invariant subspaces

Solutions of (1) can be constructed in term of the **invariant subspaces** of the following $2n \times 2n$ Hamiltonian matrix

$$\mathscr{H} := \begin{pmatrix} A & -B\,R^{-1}\,B^T \\ -Q & -A^T \end{pmatrix}$$

Theorem [Zhou et al. (1996)]

Let $\mathcal{V} \subset \mathbb{C}^{2n}$ be an n-dimensional invariant subspace of $\mathscr{H}$ and let $X_1, X_2 \in \mathbb{C}^{n \times n}$ be two complex matrices such that

$$\mathcal{V} = Im \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

If X_1 is invertible, then $X := X_2\, X_1^{-1}$ is a solution of the Riccati Equation (1).

Invariant subspaces can be obtained via eigenvalues and eigenvectors computation

The spectral factorization problem

The spectrum of $\mathscr{H}$ is **symmetric** with respect to the real and imaginary axis

If we consider the characteristic polynomial of $\mathscr{H}$

$$f(\lambda) = \det(\mathscr{H} - \lambda I_{2n})$$

Then

$$f(\lambda) = f(-\lambda)$$

Invariant subspaces can be obtained by computing factorizations of the form

$$f(\lambda) = g(\lambda)\, g(-\lambda)$$

where $g(\lambda) \in \mathbb{C}[\lambda]$

This problem is known as the **spectral factorization problem**

The problem under consideration

n^{th} order **Single Input (u) Single Output (y)** systems

$$\begin{cases} \dot{x} = A\,x + B\,u \\ y = C\,x \end{cases}$$

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \qquad B := (0 \ \dots \ 0 \ \ 1)^T \qquad C := (c_0 \quad \dots \quad c_{n-1})$$

where $a := (a_0, \cdots, a_{n-1})$, $c := (c_0, \cdots, c_{n-1})$ are **unknown parameters**.

Goal : Compute a closed loop control u that stabilizes y and minimizes

$$\frac{1}{2}\int_0^{+\infty} [y(t)^2 + u(t)^2]dt$$

This control will depend on the parameters a,c $\rightsquigarrow$ observe the effect of parameters on the optimization problem !

The problem under consideration

This yields the following Algebraic Riccati Equation

$$\mathscr{R} := X\,A + A^T\,X - X\,B\,B^T\,X + C^T\,C = 0 \quad (2)$$

where X is a symmetric matrix

Theorem [Zhou et al. (1996)]

If the pair (A, C) is observable, then

- The positive definite solution X of (2) is unique
- The positive definite solution X of (2) is a stabilizing solution

Goal : Compute the positive definite solution of (2)

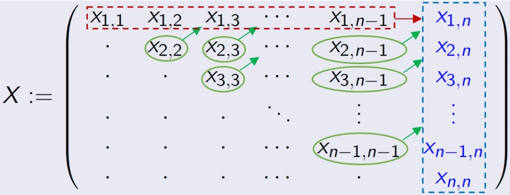
Overview

Algebraic description

$\mathscr{R} = 0 \Leftrightarrow \frac{n(n+1)}{2}$ polynomial equations of $\frac{n(n+1)}{2}$ unknowns

Noting $X = (x_{i,j})$ then $\frac{n(n-1)}{2}$ elements of $\mathscr{R}$ yields

$$x_{i,j} = x_{i-1,j+1} + f(a_k, c_k, x_{k,n} | k = 1 \cdots n)$$



Recursion $\rightarrow x_{i,j} = f(a_k, c_k, x_{k,n} | k = 1 \cdots n)$

Two halting conditions :

- Strictly above the anti-diagonal $\rightarrow$ First row
- Below the anti-diagonal $\rightarrow$ Last column

Algebraic description

For $k = 1 \dots n$, we set $x_{k,0} = x_{0,k} := 0$, and for $(i,j) \in \mathbb{N}^2$, we define :

$$\begin{cases} N(i,j) := i-1, \quad 2 \leq i+j \leq n+1 & \text{(stly. above anti-diag.)} \\ N(i,j) := n-j+1, \quad n+1 < i+j \leq 2\,n+1 & \text{(below anti-diag.)} \end{cases}$$

The elements of X solution of $\mathscr{R} = 0$ are determined only by the b_k's

$$x_{k,n} = b_{k-1} - a_{k-1} \quad \text{(last column of X)}$$

$$x_{i,j-1} = \sum_{k=0}^{N(i,j)} (-1)^k \, b_{i-1-k} \, b_{j-1+k} - \theta_{N(i,j)}$$

where $1 \leq k \leq n$, $1 \leq i < j \leq n$, and θ_m is defined by :

$$\theta_m := \sum_{k=0}^{m} (-1)^k \left(a_{i-1-k} \, a_{j-1+k} + c_{i-1-k} \, c_{j-1+k} \right)$$

The number of variables is now equal to n

A new polynomial system

Polynomial system of n equations in b_k

$$\mathcal{B} := \begin{cases} \mathcal{B}_0 := b_0^2 - d_0 = 0, \\ \mathcal{B}_k := b_k^2 + 2\displaystyle\sum_{m=1}^{M(k)} (-1)^m b_{k-m}\, b_{k+m} - d_{2k} = 0, \quad 1 \leq k \leq n-1 \end{cases}$$

where the constants d_{2k} are defined by

$$\begin{cases} d_0 := a_0^2 + c_0^2 \\ d_{2k} := 2\displaystyle\sum_{m=1}^{M(k)} (-1)^m \left(a_{k-m}\, a_{k+m} + c_{k-m}\, c_{k+m}\right) + a_k^2 + c_k^2, \\ d_{2n} := 1 \end{cases}$$

Theorem - [Rance et al. (2016)]

The polynomial system $\mathcal{B} = \{\mathcal{B}_0, \cdots, \mathcal{B}_{n-1}\}$

- is a **reduced Gröbner basis** of the ideal $\langle \mathcal{B} \rangle$ w.r.t. **the DRL order** $b_{n-1} \succ \ldots \succ b_0$
- has generically 2^n **distinct complex solutions**

Relation with the spectral factorization

Find the solutions via invariant spaces of the Hamiltonian

$$\mathscr{H} := \begin{pmatrix} A & -B\,B^T \\ -C^T\,C & -A^T \end{pmatrix} \in \mathbb{Q}(a_0, \ldots, a_{n-1}, c_0, \ldots, c_{n-1})^{2n\times 2n}.$$

$f(\lambda)$ is the characteristic polynomial of $\mathscr{H}$

$$f(\lambda) = (-1)^n \sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^k (c_l\,c_k + a_l\,a_k)\lambda^{l+k}.$$

Theorem -[Rance et al. (2016)]

Let $f(\lambda) = g(\lambda)\,g(-\lambda)$ be a factorization of f, where

$$g(\lambda) := \sum_{k=0}^{n} b_k\,\lambda^k.$$

The equations that stems from the equality are those in $\mathcal{B}$

Theorem - [Kanno et al. (2009)]

$X > 0 \Leftrightarrow \sigma := \max\{b_{n-1} \in \mathbb{R} \mid \text{solution of } \mathcal{B}\}$

Some interesting properties

Theorem - [Rance et al. (2016)]

The polynomial ideal generated by $\mathcal{B}$ is in **shape position** with respect to any variable b_{n-k} where k is odd.

Our proof is based on the spectral factorization formulation

Moreover, the system has certain symmetries that should be identified (Ongoing work)

Parametrization of the solutions of $\mathcal{B}$

Next step : parametrize the solutions of $\mathcal{B}$

- **Generically**, the system $\mathcal{B}$ can be written in the following form :

$$\begin{cases} \mathcal{P}(b_{n-1}) = 0, \\ b_{n-2} = f_{n-2}(b_{n-1}), \\ \qquad \dots \\ b_0 = f_0(b_{n-1}), \end{cases}$$

Solving $\rightsquigarrow$ computing the roots of $\mathcal{P}(b_{n-1})$ + substitution in the f_i

Study the maximum real root of $\mathcal{P}(b_{n-1})$ with respect to the parameters

Some theoretical and practical barriers

- **Size** of expressions grows exponentially
  $\rightarrow$ **limited to low order systems**
- **Degrees** of polynomials grows exponentially
  $\rightarrow$ **No closed-form solutions for high order systems**

$\Rightarrow$ Interest in **small order systems**

Overview

Small order systems - $n = 3$

$$\mathcal{B} := \begin{cases} \mathcal{B}_0 := b_0^2 - d_0 = 0 & \Rightarrow \quad b_0 \text{ completely determined} \\ \mathcal{B}_1 := b_1^2 - 2\, b_0\, b_2 - d_2 = 0 & \\ \mathcal{B}_2 := b_2^2 - 2\, b_1 - d_4 = 0 & \Rightarrow \quad b_1 = \frac{1}{2}\,(b_2^2 - d_4) \end{cases}$$

$\mathcal{B}_1 \Rightarrow$ univariate polynomial $\mathcal{P}$ in b_2

$$\mathcal{P}(b_2) := b_2^4 - 2\, d_4\, b_2^2 - 8\, b_0\, b_2 + d_4^2 - 4\, d_2 = 0$$

Its roots are $b_2(\varepsilon) = \varepsilon_1\, \frac{1}{2}\, \sqrt{2\, u} + \varepsilon_2\, \frac{1}{2}\, \sqrt{\Delta_2}$ with

$$\begin{cases} \varepsilon_1 := \pm 1,\ \varepsilon_2 := \pm 1,\ \varepsilon := (\varepsilon_1, \varepsilon_2), \\ p_2 := 4\, d_2 - \frac{4}{3}\, d_4^2, \\ q_2 := \frac{8}{3}\, d_2\, d_4 - \frac{16}{27}\, d_4^3 - 8\, b_0^2, \end{cases} \quad \begin{cases} \alpha := \left(\frac{-27\, q_2 + \sqrt{27\left(4\, p_2^3 + 27\, q_2^2\right)}}{2} \right)^{1/3} \\ u := \frac{1}{3}\left(\alpha - \frac{3\, p_2}{\alpha} + 2\, d_4 \right), \\ \Delta_2 := 2\left(2\, d_4 + \varepsilon_1\, \frac{8\, b_0}{\sqrt{2\, u}} - u \right). \end{cases}$$

Determine $X > 0$: which root is the greatest ?

Discriminants of univariate polynomials

Choosing $b_{2,\max}$ $\Leftrightarrow$ Computing the **discriminant** of $\mathcal{P}$

Discriminant of a quadratic polynomial

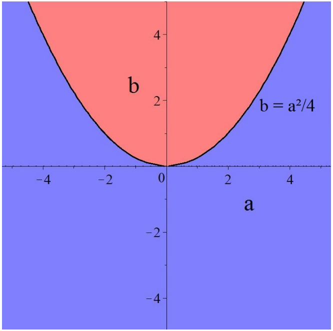
Let $P(x) = x^2 + a\,x + b$, $x \in \mathbb{R}$, $(a, b) \in \mathbb{R}^2$.
The discriminant of P is defined by :

$$\Delta(a, b) = a^2 - 4\,b.$$

Roots of P :

$$x_{1,2} = -a \pm \sqrt{\Delta(a, b)}.$$

$\Rightarrow$ **When $\Delta(a, b) = 0$, x_1 and x_2 are crossing !**

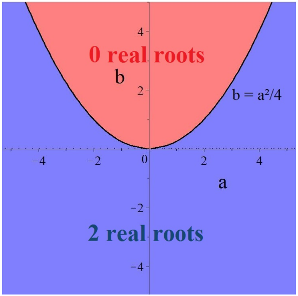


Using the **Discriminants** of univariate polynomials

Exemple of a second order polynomial

$$P(x) = x^2 + a\,x + b = 0$$
$$\mathrm{disc_x(P) = a^2 - 4\,b}$$
$$x_{1,2}(a,b) = -a \pm \sqrt{\mathrm{disc_x(P)}}$$

Red cell : $x_{1,2}(0,1) = \pm 2\,i \in \mathbb{C}$
Blue cell : $x_{1,2}(0,-1) = \pm 2 \in \mathbb{R}$



Discriminant of

$$\mathcal{P}(b_2) := b_2^4 - 2\,d_4\,b_2^2 - 8\,b_0\,b_2 + d_4^2 - 4\,d_2$$

We apply the same reasoning to $\mathcal{P}(b_2)$ to prove that

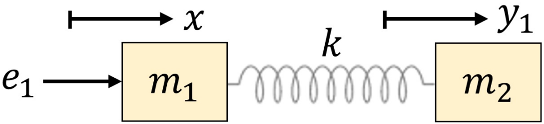
$$\sigma = \frac{1}{2}\sqrt{2\,u} + \frac{1}{2}\sqrt{\Delta_2}$$

is the maximal real root of $\mathcal{P}$ for any values of the parameters.

Overview

A practical example

Two-mass-spring system :



$$c := \frac{k}{m_1\, m_2}$$

$$a_2 := \frac{m_1 + m_2}{m_1\, m_2}\, k$$

$$G := \frac{y_1}{e_1} = \frac{c_0}{s^2(s^2 + a_2)}$$

$$X := \begin{pmatrix} b_0\, b_1 & b_0\, b_2 & b_0\, b_3 & b_0 \\ b_0\, b_2 & b_1\, b_2 - b_0\, b_3 & b_1\, b_3 - b_0 & b_1 \\ b_0\, b_3 & b_1\, b_3 - b_0 & b_2\, b_3 - b_1 & b_2 - a_2 \\ b_0 & b_1 & b_2 - a_2 & b_3 \end{pmatrix}$$

$$\mathcal{B} \Rightarrow \begin{cases} \mathcal{B}_0 := b_0^2 - c_0^2 = 0 \\ \mathcal{B}_1 := b_1^2 - 2\, b_0\, b_2 = 0 \\ \mathcal{B}_2 := b_2^2 - 2\, b_1\, b_3 + 2\, b_0 - a_2^2 = 0 \\ \mathcal{B}_3 := b_3^2 - 2\, b_2 + 2\, a_2 = 0 \end{cases}$$

A practical example

- A parametrization of $\mathcal{B}$ is easily found :

$$\begin{cases} b_0 = c_0 \\ b_1 = \dfrac{b_3^4 + 4\,a_2\,b_3^2 + 8\,c_0}{8\,b_3} \\ b_2 = \frac{1}{2}\,b_3^2 + a_2 \\ \mathcal{P}(b_3) := b_3^8 + 8\,a_2\,b_3^6 + 16\,(a_2^2 - 3\,c_0)\,b_3^4 - 64\,a_2\,c_0\,b_3^2 + 64\,c_0^2 = 0 \end{cases}$$

- $\mathcal{P}$ is of degree 4 $\Rightarrow$ symbolic
- Positive definite solution is given by $X(\sigma)$ where :

$$\sigma := \sqrt{2}\sqrt{\left(\sqrt{2\,c_0} - a_2\right) + \sqrt{\left(\sqrt{2\,c_0} - a_2\right)^2 + 2\,c_0}}$$

- In this case : $X(c_0, a_2)$

Overview

Conclusion and perspectives

- Contributions :
  - Symbolic techniques in automatic control problems to handle parameters
  - Closed form control with respect to the parameters
  - Also used for H_∞ control
- Ongoing work :
  - Study the symmetries of the systems that stem from the Riccati Equations
  - Extension to higher order systems $\rightsquigarrow$ work with implicit equations
  - Extension to **MIMO** systems

Thank you for your attention

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