Regularity and Gröbner bases of the Rees algebra of edge ideals of bipartite graphs

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Let $G = (V(G), E(G))$ be a bipartite graph on the vertex set $V(G) = X \cup Y$ with bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Let $\mathbb{K}$ be a field and let R be the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. The edge ideal $I = I(G)$, associated to G, is the ideal of R generated by the set of monomials $x_i y_j$ such that x_i is adjacent to y_j.

We study several properties of the Rees algebra of I. From a computational point of view we first focus on the universal Gröbner basis of its defining equations, and from a more algebraic standpoint we focus on its total and partial regularities as a bigraded algebra. Applying these ideas, we give an estimation of when $\mathrm{reg}(I^s)$ starts to be a linear function and we find upper bounds for the regularity of the powers of the edge ideal I.

Let $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^i t^i \subset R[t]$ be the Rees algebra of the edge ideal I. Let $f_1, \ldots, f_q$ be the square free monomials of degree two generating I. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, \ldots, T_q]$ via the map

$$\begin{aligned} S = \mathbb{K}[x_1, \ldots, x_n, y_1 \ldots, y_m, T_1, \ldots, T_q] &\xrightarrow{\psi} \mathcal{R}(I) \subset R[t], \\ \psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_i) &= f_i t. \end{aligned}$$

Then the presentation of $\mathcal{R}(I)$ is given by $S/\mathcal{K}$ where $\mathcal{K} = \mathrm{Ker}(\psi)$.

The universal Gröbner basis of the ideal $\mathcal{K}$ is defined as the union of all the reduced Gröbner bases $\mathcal{G}_<$ of the ideal $\mathcal{K}$ as $<$ runs over all possible monomial orders ([1]). In our first main result we compute the universal Gröbner basis of the defining equations $\mathcal{K}$ of the Rees algebra $\mathcal{R}(I)$.

Theorem 1. *Let G be a bipartite graph and $\mathcal{K}$ be the defining equations of the Rees algebra $\mathcal{R}(I(G))$. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ is given by*

$$\begin{aligned} \mathcal{U} = &\{T_w \mid w \textit{ is an even cycle}\} \\ &\cup \{v_0 T_{w^+} - v_a T_{w^-} \mid w \textit{ is an even path}\} \\ &\cup \{u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \mid w_1 \textit{ and } w_2 \textit{ are disjoint odd paths}\}. \end{aligned}$$

Our second main result is computing the total regularity and giving upper bounds for both partial regularities ([2]) of $\mathcal{R}(I)$ as a bigraded S-algebra.

Theorem 2. *Let G be a bipartite graph. Then we have:*

(i) $\mathrm{reg}(\mathcal{R}(I(G))) = \mathrm{match}(G)$,

(ii) $\mathrm{reg}_{xy}(\mathcal{R}(I(G))) \leq \mathrm{match}(G) - 1$,

(iii) $\operatorname{reg}_T(\mathcal{R}(I(G))) \leq \operatorname{match}(G)$,

where match(G) denotes the matching number of G.

It is a famous result (for a general ideal in a polynomial ring) the asymptotic linearity of $\operatorname{reg}(I^s)$ for $s \gg 0$ ([3]). However, the exact form of this linear function and the exact point where $\operatorname{reg}(I^s)$ starts to be linear, is a problem that continues wide open even in the case of monomial ideals. In recent years, a number of researchers have focused on computing the regularity of powers of edge ideals and on relating these values to combinatorial invariants of the graph.

From the characterization of the universal Gröbner basis and a special monomial order, we get the following results.

Corollary 3. *Let G be a bipartite graph with bipartition $V(G) = X \cup Y$. Then, for all $s \geq 1$ we have*

$$\operatorname{reg}(I(G)^s) \leq 2s + \min\left\{|X| - 1,\ |Y| - 1,\ 2b(G) - 1\right\},$$

where $b(G)$ represents the minimum cardinality of the maximal matchings of G. In the particular case of G being a complete bipartite graph we have

$$\operatorname{reg}(I(G)^s) = 2s.$$

Using the existence of the canonical module for the Rees algebra, we can obtain our last results.

Corollary 4. *Let G be a bipartite graph. Then, the following statements hold:*

(i) For all $s \geq \operatorname{match}(G) + |E(G)| + 1$ we have

$$\operatorname{reg}(I(G)^{s+1}) = \operatorname{reg}(I(G)^s) + 2.$$

(ii) For all $s \geq 1$ we have

$$\operatorname{reg}(I(G)^s) \leq 2s + \operatorname{match}(G) - 1.$$

References

[1] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

[2] Tim Römer, *Homological properties of bigraded algebras*, IIllinois J. Math. 45 (2001), no. 4, 1361-1376.

[3] S. Dale Cutkosky, Jürgen Herzog, and Ngô Viêt Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math. 118 (1999), no. 3, 243 - 261.